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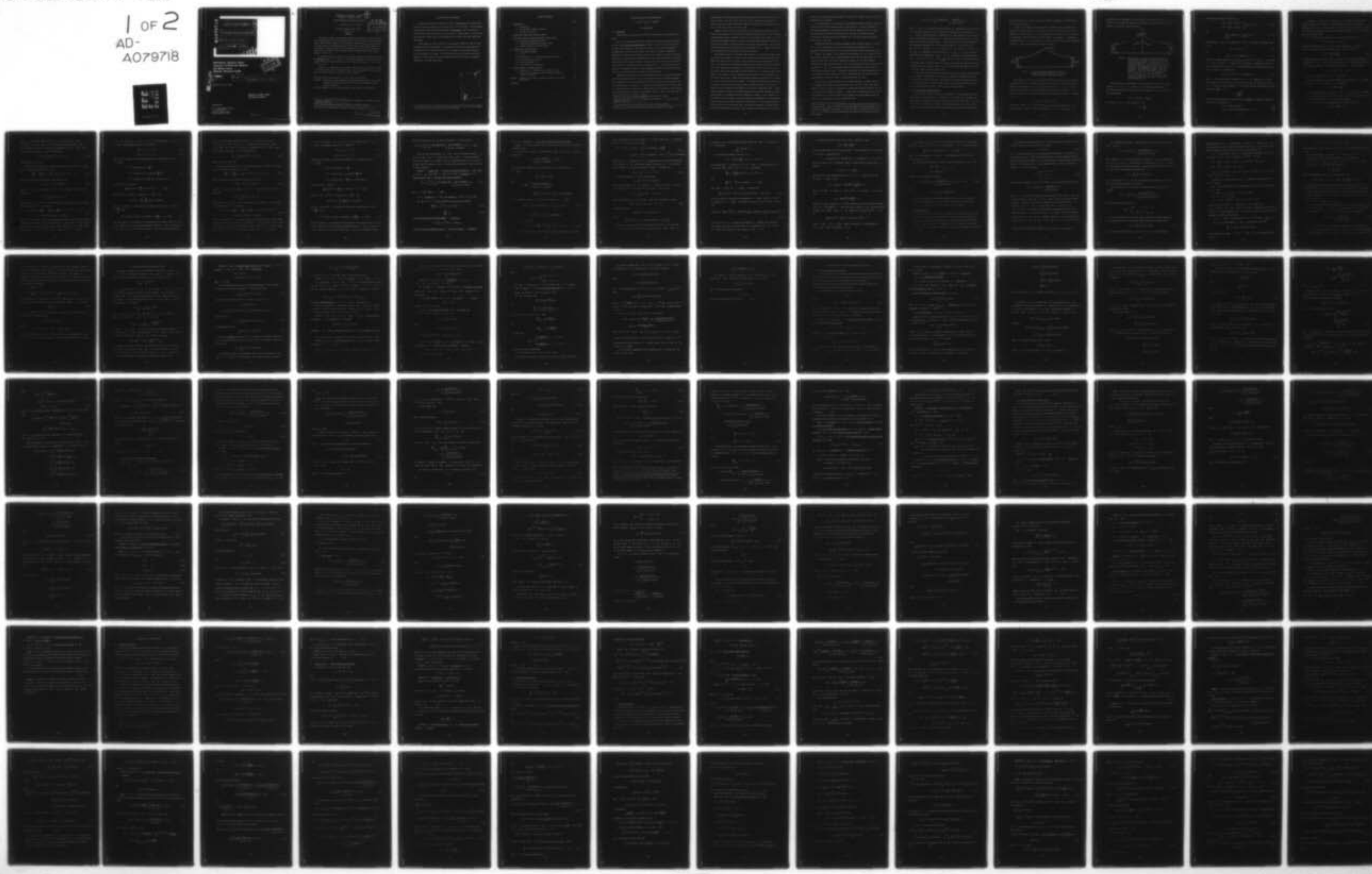

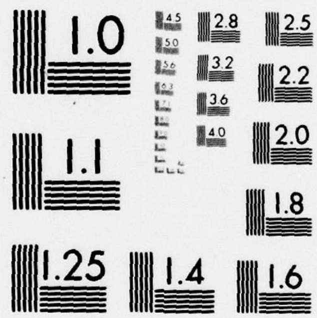

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(9) MRC Technical Summary Rep[redacted]t #2012 (13) (3)

(6) FINITE-AMPLITUDE SOLITARY WATER WAVES.

(14) MRC-TSR-2φ12

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(12) 118

(15) DAAG29-75-C-ØØ24

(Received July 25, 1979)

Sponsored by

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Research Triangle Park
North Carolina 27709

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UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

FINITE-AMPLITUDE SOLITARY WATER WAVES

C. J. Amick[†] and J. F. Toland[††]

Technical Summary Report #2012

October 1979

ABSTRACT

This paper considers the existence problem for solutions of the free boundary-value problem which arises from the question of the existence of solitary gravity waves, moving without change of form, and with constant velocity, on the surface of ideal fluid in a horizontal canal of finite depth. The analysis imposes no restriction on either the slope or the amplitude of the wave, and we prove that there exists a connected set of solitary waves containing waves of all slopes between 0 and $\pi/6$.

It is then proved that each of these solitary waves has finite mass, and, as a consequence, that $F > 1$, where F is the Froude number. This, in turn, tells us that the solitary wave decays faster than $\exp(-\alpha|\frac{x}{h}|)$, where $\alpha \in [0,\tilde{\alpha})$ and $\tilde{\alpha}^{-1} \tan \tilde{\alpha} = F^2$.

Finally, it is shown that, in a certain limit, these solitary waves converge to a solitary Stokes wave of greatest height, and the validity of Stokes' conjecture for solitary waves is considered, but not resolved.

AMS (MOS) Subject Classifications: 76.45, 45G05, 45C05, 47H15.

Key Words: Nekrassov's Integral Equation; global existence theory; Froude number; asymptotic decay; Stokes wave and Stokes wave of greatest height; Stokes' conjecture.

Work Unit Numbers 1 and 3 - Applied Analysis; Applications of Mathematics

[†]Research supported by a N.A.T.O. Postdoctoral Fellowship. St. John's College, Cambridge, England.

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Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

SIGNIFICANCE AND EXPLANATION

Recently the problem of computing the shape of large amplitude gravity water-waves has enjoyed a great deal of success ([11], [13], [31]-[33]). The corresponding existence questions have been answered 'in the large' only in the case of periodic water waves, and not as yet in the case of solitary waves. In the latter case existence results have so far pertained only to waves with small amplitudes and slopes.

In this paper we attack, and solve, the existence problem for solitary waves without restriction on amplitude. The methods used involve the applications of the modern theory of global bifurcation to a sequence of approximate problems, and then passing to the limit. The conclusions drawn give rigorous theoretical support to results which had been previously derived as the result of numerical computation, or formal calculation.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

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FINITE-AMPLITUDE SOLITARY WATER WAVES

C. J. Amick[†] and J. F. Toland[††]

1. INTRODUCTION

1.1. Background

In describing the solitary wave on water of finite depth, Lord Rayleigh [40] wrote:

"This is the name given by Mr. Scott Russell[*] to a particular wave described by him in the British Association Report of 1844. Since its length is about six or eight times the depth of the canal, this wave is, to a rough approximation, included under the theory of long waves; but there are several circumstances observed by Mr. Russell which indicate that it has a character distinct from other long waves. Among these may be mentioned the very different behaviour of solitary waves according as they be positive or negative, viz. according as they consist of an elevation or a depression from the undisturbed level. In the former case the wave has a remarkable permanence, being propagated to great distances without much loss; but a negative wave is soon broken up and dissipated."

Regarding the problem as one of steady motion, he gave a theoretical explanation of this behaviour, using a series expansion method which depends ultimately on an assumption that the wave amplitude is small. Independently, Boussinesq [11] had reached similar conclusions. The equation of Kortewig and de Vries [21], which is a model equation for long waves in shallow water, has a family of exact solitary wave solutions which are known in closed form. There are many other situations in

[*] see [41], and [35] for further remarks on Scott Russell's report.

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hydrodynamics where model equations may be expected to possess solitary wave solutions, [3]-[6], [15], [27], [37], [38], [45], [46], and recently considerable interest has been focused on questions about the existence of such solutions, [2], [5], [7]-[10], [16], [26], [37], [45], [46].

However, much of the work on the theory of solitary waves on water of finite depth has been devoted to the problem of estimating the wave profile. Most recently, Longuet-Higgins and his colleagues, Fenton, Fox, Byatt-Smith and Cokelet [12], [13], [30]-[32], have combined physical insight with sophisticated numerical techniques to calculate the profile of very high waves, both solitary and periodic. Their results show great stability and consistency, and the problem of how to compute the shape of large amplitude water-waves appears to have been solved.

The existence theory for solitary wave solutions of the exact water-wave problem is much less complete. Krasovskii [24] claims that solitary waves with Froude number less than unity do not exist, and that solitary waves of depression do not exist. The question of existence was addressed by Lavrentiev [26] and Ter-Krikorov [45], [46], who proved the existence of infinitesimal solitary waves as the limit of periodic waves whose wavelength increases indefinitely. A second and more direct proof of the existence of small amplitude solitary waves is due to Fredrichs and Hyers [16], who justify a power series expansion method by proving that it converges to a non-trivial solution of the problem. (The approach of Fredrichs and Hyers has been given a brief, but rather sophisticated treatment by Beale [2], who uses an implicit function theorem of Nash-Moser type to obtain an improved result.) These authors consider the exact free surface boundary value problem, and make no approximation based on the assumption that the amplitude is small. It is, therefore, to be expected that the equations also have solutions corresponding to large amplitude solitary waves, such as those observed by Scott Russell. In the present paper we will show that this is indeed the case. We prove the existence of solitary water waves of all amplitudes, from zero to and including the solitary wave of greatest

height, each of which has Froude number (see section 4.5) greater than one, and decays exponentially at infinity.

It is appropriate, at this stage, to say something about the periodic water-wave problem, where a global existence theory already exists. Using the formulation of the periodic problem as a non-linear integral equation (Nekrassov [36]), a global existence theory was found by Krasovskii [23], who used the monotone minorant theorem [22], and later by Keady and Norbury [19], who used a global bifurcation theorem [14], [39], [48].

Krasovskii proved the following remarkable global existence theorem: *for each* β, $0 \leq \beta < \frac{\pi}{6}$, *there exists a periodic water-wave whose maximum slope is* β. Krasovskii's method enabled him to say nothing about waves with slope greater than $\frac{\pi}{6}$.* However, this number $\frac{\pi}{6}$ was suggestive since Stokes [43; vol 1, page 227] had argued that a periodic wave of greatest height exists and is sharp-created, the tangents on either side of the crest subtending an angle $\frac{\pi}{6}$ with the horizontal. Krasovskii therefore conjectured that a wave with slope greater than $\frac{\pi}{6}$ does not exist, and that as $\beta \to \frac{\pi}{6}$, the corresponding periodic wave converges to the so-called Stokes wave of greatest height. (Numerical calculations of Longuet-Higgins and Fox [31] indicate that this conjecture is false, and McLeod [33] has recently found a rigorous proof to show that waves of slope greater than $\frac{\pi}{6}$ exist.)

More recently, Keady and Norbury [19] examined Nekrassov's integral equation as an example where global bifurcation theory (see, e.g. [14], [39], [48]) may be applied. They use a version of Nekrassov's equation which differs from the one used by Krasovskii by a change of variables. For the case of periodic waves on a flow of infinite depth it is rather similar to the equation for solitary waves derived in section 1.2, and we give it here for comparison [34]:

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*The restriction $0 \leq \beta < \frac{\pi}{6}$ arises from the following technical considerations: in the Banach space $C_0[0,\pi]$, the Nekrassov integral equation used by Krasovskii can be proved to be completely continuous, in a ball of radius β about the origin, only if $\beta < \frac{\pi}{6}$. The theory of positive operators [22] then fails to prove more than is stated above.

$$\theta(s) = \frac{2}{3}\int_0^{\pi} \frac{1}{\pi}\left\{\sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k}\right\} \frac{\sin\theta(t)}{\frac{1}{\mu} + \int_0^t \sin\theta(w)\,dw}\,dt \quad . \tag{1.1}$$

Here $s \in [-\pi,\pi]$ is a function of position along one wavelength of the free surface, θ is the angle between the free surface and the horizontal at that point, and $\mu > 0$. Since a similar equation is examined in the next section, we will not dwell here on the physical significance of μ, except to say that the case $\frac{1}{\mu} = 0$ corresponds to the wave of greatest height. Keady and Norbury [19] show the existence of an unbounded connected set of non-trivial solutions, (μ,θ) of (1.1), in $\mathbb{R} \times C_0[-\pi,\pi]$, such that $\mu > 3$, θ is odd, and $0 < \theta(s) < \frac{\pi}{2}$ for $s \in (0,\pi)$. If $\{(\mu_n,\theta_n)\}$ is a sequence of such solutions of (1.1), and $\mu_n \to \infty$, then it can be shown [47] that a subsequence $\{\theta_{n(k)}\}$ of $\{\theta_n\}$ converges to a non-trivial solution θ of (1.1) with $\frac{1}{\mu} = 0$, and $\limsup_{s\to 0+} \theta(s) \geq \frac{\pi}{6}$. Thus, the existence question for periodic waves of all amplitudes up to and including the periodic wave of greatest height has been settled, though it has yet to be shown that Stoke's conjecture is true for a wave of greatest height [47].

In the next section we derive an integral equation for solitary waves analogous to (1.1) above. Unfortunately, the equation is singular, and global bifurcation theory may not be applied directly. Nevertheless, by methods explained in section 1.3, we can prove that a global existence theory for solitary water-waves exists. and is in many respects similar to that for the periodic problem.

1.2. The nonlinear integral equation

We shall show in this section that the existence of finite-amplitude solitary waves follows from the existence of solutions 'in the large' of an integral equation analogous to Nekrassov's integral equation for periodic waves [36]. This integral equation resembles one introduced by Milne-Thomson [34; pp. 416-422], but it differs in that the variables involved have different interpretations physically.

The derivation here is similar, but in what follows we emphasize its implications for an existence theory.

Consider a two-dimensional solitary wave of elevation and of permanent form moving with constant velocity c from left to right on the surface of an incompressible, irrotational flow. It will be assumed that the bottom of the flow domain is horizontal, and that the pressure at the surface is atmospheric. Furthermore, we assume that as $x \to \pm\infty$, the flow is at rest, and that the free surface approaches a height h above the bottom of the channel (Figure 1).

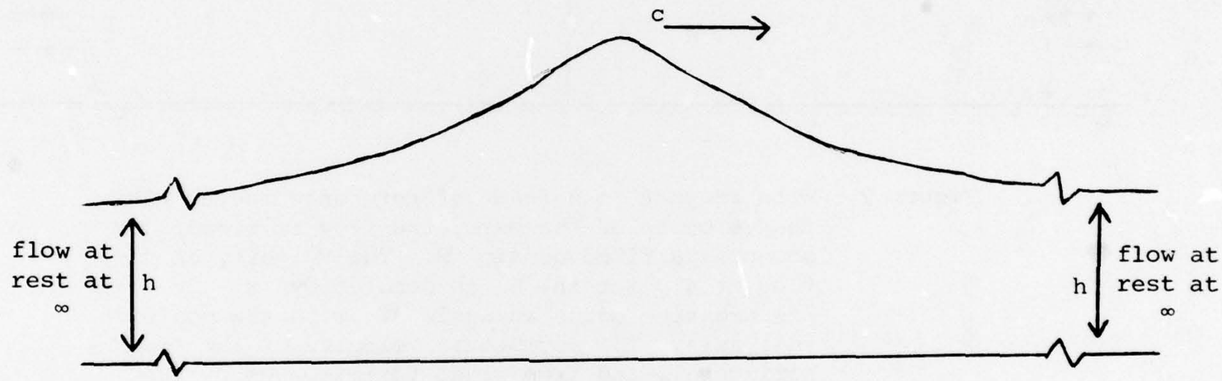

Figure 1. Solitary wave whose asymptotic height is h moving from left to right on the surface of an ideal liquid in a horizontal channel.

With respect to a frame of reference moving with the velocity of the solitary wave profile, the flow is steady and occupies a fixed region. Moreover, the free surface is stationary and the velocity of the flow as $x \to \pm\infty$ is $-c$. In this moving reference frame, the steady irrotational flow occupies a domain $S \subset \mathbb{C}$, which lies between the real axis and some a priori unknown free surface Γ (Figure 2). Let

$$\Gamma = \{(x,y) : y = H(x), x \in \mathbb{R}\} \quad .$$

Clearly the function H is an even function, is strictly decreasing for $x > 0$, and approaches a height h above the real axis as $x \to \pm\infty$. The profile Γ is

determined by the requirement that the pressure in the flow be constant on the free surface. We now reduce the question of existence for this steady free surface problem to a nonlinear integral equation.

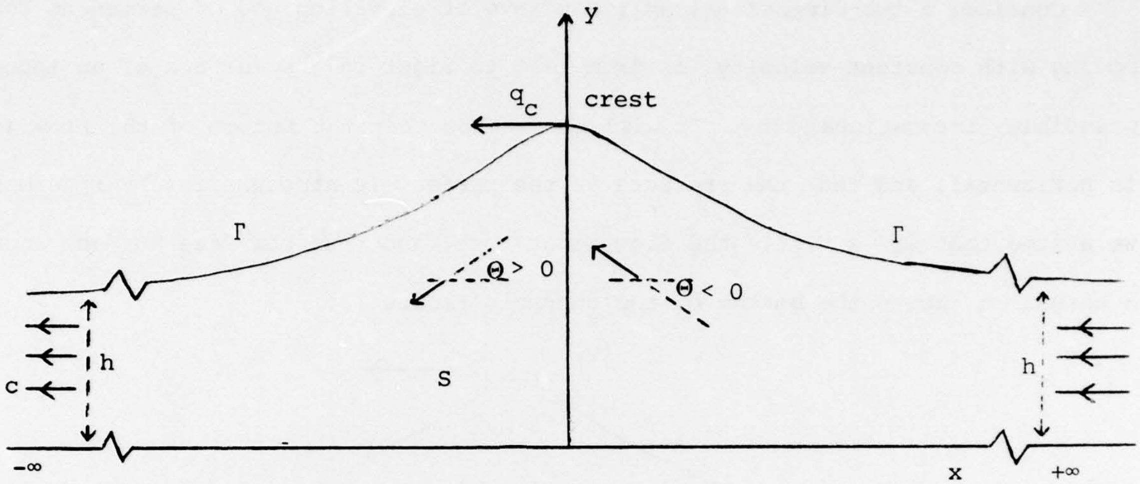

Figure 2. With respect to a frame of reference moving with the velocity of the wave, the flow is steady and occupies a fixed domain S. The velocity of the flow at a point in S is denoted by $u + iv$, and its negative makes an angle Θ with the positive real axis. The asymptotic velocity of the flow is horizontal, and from right to left. Under the mapping m, the crest corresponds to $e^{it} \in \partial\mathcal{D}$ with $t = 0$, and Γ at $x = \pm\infty$ corresponds to $e^{it} \in \partial\mathcal{D}$ with $t = \mp \pi/2$. Thus $\Theta(t) > 0$, $t \in (0,\pi/2)$ and $\Theta(t) < 0$, $t \in (-\pi/2,0)$. The velocity of the flow at the crest is horizontal, and has magnitude q_c.

Since the flow is incompressible and irrotational, there exists a complex potential $\tilde{w} = \tilde{\phi} + i\tilde{\psi}$ in S, where $\tilde{\phi}$ is the velocity potential and $\tilde{\psi}$ is the Stokes stream function; more precisely, the velocity (u,v) of the flow has the representation

$$(u,v) = (-\tilde{\phi}_x,-\tilde{\phi}_y) = (-\tilde{\psi}_y,\tilde{\psi}_x) \quad .$$

The velocity (u,v) is then related to $\tilde{w}$ by

$$u - iv = -\frac{d\tilde{w}}{dz} \quad . \tag{1.2}$$

The boundary conditions at infinity are

$$\left.\begin{aligned} u(z) &= -\tilde{\phi}_x(z) = -\tilde{\psi}_y(z) \to -c \\ v(z) &= -\tilde{\phi}_y(z) = \tilde{\psi}_x(z) \to 0 \end{aligned}\right\} \text{ as } |z| \to \infty , \tag{1.3}$$

and

$$\lim_{\substack{|z|\to\infty, \\ z\in\Gamma}} \text{Imaginary } z = \lim_{|x|\to\infty} H(x) = h . \tag{1.4}$$

The functions $\tilde{\phi}$ and $\tilde{\psi}$ are harmonic in S, and $\tilde{\psi}$ satisfies the boundary conditions

$$\tilde{\psi}(z) = ch \quad \text{for} \quad z \in \Gamma , \tag{1.5}$$

and

$$\tilde{\psi}(z) = 0 \quad \text{for} \quad \text{Imaginary } z = 0 . \tag{1.6}$$

Indeed, the normal component of the velocity is zero on ∂S, and so $\tilde{\psi}$ is constant on Γ and on $\{\text{Imag } z = 0\}$. Equation (1.6) represents a normalization, and (1.5) then follows from (1.3) and (1.4). The coupling between the flow and its free surface is given by the condition that

$$\frac{1}{2} u^2(z) + \frac{1}{2} v^2(z) + gy = \text{constant}, \ z \in \Gamma , \tag{1.7}$$

where g is the gravitational constant and $z = x + iy$. Equation (1.7) follows from Bernoulli's theorem [34, pp. 9-11] and the fact that the pressure is constant on Γ. In terms of these parameters, it is usual to define a dimensionless quantity, the Froude number, F by

$$F = \left(\frac{c^2}{gh}\right)^{1/2} .$$

Since this paper deals with a solitary wave of <u>elevation</u>, we impose the following conditions on the function H:

$$H \text{ is an even function,} \tag{1.8a}$$

$$H(x) \text{ is strictly decreasing for } x > 0. \tag{1.8b}$$

In Theorem 1.1, we show that the existence of a solution of the nonlinear integral equation (1.17) leads to a complex potential $\tilde{w}$ and a free surface Γ satisfying (1.3)-(1.8). In the light of this fact, the rest of the paper is devoted to a study of this equation. Before giving a proof of Theorem 1.1, we first show how (1.17) actually arises.

Let $\mathcal{D}$ denote the open unit disc in the complex ζ-plane and let

$$\mathcal{D}' = \bar{\mathcal{D}} \cap \{\text{Real } \zeta \geq 0,\ \zeta \neq \pm i\} \quad .$$

Following Milne-Thomson [34, p. 417], we seek a conformal map m from $\mathcal{D}'$ onto $\bar{S}$ such that the complex potential $\tilde{w}(z) = w(m^{-1}(z))$, where

$$w(\zeta) = \frac{-2ch}{\pi} \ell n\left(\frac{i+\zeta}{i-\zeta}\right),\ \zeta \in \mathcal{D}' \quad , \tag{1.9}$$

determines a flow in the z-plane which satisfies all the solitary wave boundary conditions (1.3)-(1.8). (We take the usual branch of ℓn which is defined except on the negative real axis.) The solitary wave profile Γ is then given by

$$\Gamma = \{z : z = m(e^{it}),\ -\frac{\pi}{2} < t < \frac{\pi}{2}\} \quad ,$$

here $t = \pm \frac{\pi}{2}$ correspond to $x = \mp\infty$, respectively.

The trivial example of uniform horizontal flow with velocity $-c$ everywhere in a region $S = \{z : 0 < \text{Imag } z < h\}$ is generated by

$$m(\zeta) = \frac{-2h}{\pi} \ell n\left(\frac{i+\zeta}{i-\zeta}\right) \quad ,$$

and the corresponding complex potential is $\tilde{w}(z) = cz$. The function $\tilde{w}$ satisfies all the boundary conditions except (1.8b). Since $c,h > 0$ are arbitrary in this example, the Froude number F can be any positive number.

The idea now is to seek <u>non-trivial</u> solitary waves by putting

$$m(\zeta) = \frac{-2h}{\pi}\{\ell n\left(\frac{i+\zeta}{i-\zeta}\right) + f(\zeta)\} \quad ,$$

where f is an analytic function on $\bar{\mathcal{D}} \setminus \{\pm i\}$. We require that m map $\mathcal{D} \cap \{\text{Real } \zeta = 0\}$ onto $\{\text{Imag } z = 0\}$, the bottom of the flow domain, whence $f(-\bar{\zeta}) = \overline{f(\zeta)}$. For (1.8a) to be satisfied, we demand that $f(\bar{\zeta}) = -\overline{f(\zeta)}$. These two conditions imply that f has a representation

$$f(\zeta) = \sum_{k=0}^{\infty} a_k (i\zeta)^{2k+1}, \ \zeta \in \bar{\mathcal{D}} \setminus \{\pm i\} \quad , \tag{1.10}$$

where all the a_k are real.

Since the complex potential is to have the form $\tilde{w}(z) = w(m^{-1}(z))$, the function m must be invertible on $\mathcal{D}$, and so

$$\frac{dm}{d\zeta}(\zeta) = \frac{4hi}{\pi(1+\zeta^2)} \{1 + \frac{i}{2}(1 + \zeta^2) f'(\zeta)\} \neq 0 \quad \text{for} \quad \zeta \in \mathcal{D} \quad . \tag{1.11}$$

Since the terms within the brackets cannot vanish, we set

$$1 + \frac{i}{2}(1 + \zeta^2) f'(\zeta) = \exp(\tilde{\Omega}(\zeta)) \quad , \tag{1.12}$$

where $\tilde{\Omega} = \tilde{T} + i\tilde{\Theta}$ is to be analytic on $\bar{\mathcal{D}} \setminus \{\pm i\}$. Note that (1.10) and (1.12) imply that

$$\tilde{\Omega}(\zeta) = \sum_{k=0}^{\infty} b_k \zeta^{2k}, \ z \in \bar{\mathcal{D}} \setminus \{\pm i\} \quad ,$$

where all the b_k are real. The transformation (1.12) enables us to give a physical interpretation to $\tilde{\Omega}$ as follows:

$$\begin{aligned}(u - iv)(z) &= -\frac{d\tilde{w}}{dz}(z) = -\frac{dw}{d\zeta}(m(\zeta)) \Big/ \frac{dm}{d\zeta}(\zeta) = -c\{1 + \frac{i}{2}(1 + \zeta^2) f'(\zeta)\}^{-1} \\ &= -c \exp(-\tilde{T}(\zeta))\{\cos \tilde{\Theta}(\zeta) - i \sin \tilde{\Theta}(\zeta)\} \quad .\end{aligned} \tag{1.13}$$

Hence, $\tilde{\Theta}(\zeta) = \tilde{\Theta}(m^{-1}(z))$ is the angle between the negative velocity vector and the real axis in the z-plane, while $c \exp(-\tilde{T}(m^{-1}(z)))$ is the speed of the flow. Since (1.8a) is to be satisfied, it follows that $\Theta(t) = -\Theta(-t)$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, where $\Theta(t) = \tilde{\Theta}(e^{it})$ and $T(t) = \tilde{T}(e^{it})$. Since m takes $\{e^{it} : 0 < t < \frac{\pi}{2}\}$ onto

$\Gamma \cap \{x < 0\}$, and since $v < 0$ on $\Gamma \cap \{x < 0\}$, it follows that $0 < \Theta(t) < \frac{\pi}{2}$ on $(0,\frac{\pi}{2})$. The boundary condition (1.3) follows if

$$T(t) \to 0 \quad \text{as} \quad t \to \pm \frac{\pi}{2} \ , \tag{1.14a}$$

$$\Theta(t) \to 0 \quad \text{as} \quad t \to \pm \frac{\pi}{2} \ . \tag{1.14b}$$

With all this in mind, the free surface condition (1.7) now means that for $t \in (-\frac{\pi}{2},\frac{\pi}{2})$:

$$\begin{aligned} 0 &= -c^2\, T'(t)\exp(-2T(t)) + g\,\frac{dy}{dt} \\ &= -c^2\, T'(t)\exp(-2T(t)) + g\,\mathrm{Imag}(ie^{it}\,\frac{dm}{d\zeta}(e^{it})) \qquad (1.15) \\ &= -c^2\, T'(t)\exp(-2T(t)) - \frac{2gh}{\pi}\sec t\,\sin\Theta(t)\exp(T(t)) \end{aligned}$$

by (1.11)-(1.13). Therefore

$$\frac{d}{dt}(\exp(-3T(t)) - \frac{6gh}{\pi c^2}\sec t\,\sin\Theta(t) = 0, \quad t \in (-\frac{\pi}{2},\frac{\pi}{2}) \ ,$$

and so if $\sec t\,\sin\Theta(t)$ is integrable on $(-\frac{\pi}{2},\frac{\pi}{2})$, then

$$\exp(-3T(t)) = \frac{q_c^3}{c^3} + \frac{6gh}{\pi c^2}\int_0^t \sec w\,\sin\Theta(w)\,dw \ ,$$

where $q_c = c\exp(-T(0))$ is the speed of the flow at the wave crest. Setting $\mu = \frac{6ghc}{\pi q_c^3}$ gives

$$-T(t) = \frac{1}{3}\{\ln(1/\mu + \int_0^t \sec w\,\sin\Theta(w)\,dw) + \ln(\frac{6gh}{\pi c^2})\}, \quad t \in (-\frac{\pi}{2},\frac{\pi}{2}) \ . \tag{1.16}$$

This last equation is an expression of the Bernoulli free surface condition in terms of the boundary value of the conjugate harmonic functions $\tilde{T}$ and $\tilde{\Theta}$ introduced in (1.12). The use of (1.16) with the fact that $\tilde{T}$ and $\tilde{\Theta}$ are conjugate harmonic

where f is an analytic function on $\bar{\mathcal{D}} \setminus \{\pm i\}$. We require that m map $\mathcal{D} \cap \{\text{Real } \zeta = 0\}$ onto $\{\text{Imag } z = 0\}$, the bottom of the flow domain, whence $f(-\bar{\zeta}) = \overline{f(\zeta)}$. For (1.8a) to be satisfied, we demand that $f(\bar{\zeta}) = -\overline{f(\zeta)}$. These two conditions imply that f has a representation

$$f(\zeta) = \sum_{k=0}^{\infty} a_k (i\zeta)^{2k+1}, \quad \zeta \in \bar{\mathcal{D}} \setminus \{\pm i\} \quad , \tag{1.10}$$

where all the a_k are real.

Since the complex potential is to have the form $\tilde{w}(z) = w(m^{-1}(z))$, the function m must be invertible on $\mathcal{D}$, and so

$$\frac{dm}{d\zeta}(\zeta) = \frac{4hi}{\pi(1+\zeta^2)} \{1 + \frac{i}{2}(1 + \zeta^2)f'(\zeta)\} \neq 0 \quad \text{for} \quad \zeta \in \mathcal{D} \quad . \tag{1.11}$$

Since the terms within the brackets cannot vanish, we set

$$1 + \frac{i}{2}(1 + \zeta^2)f'(\zeta) = \exp(\tilde{\Omega}(\zeta)) \quad , \tag{1.12}$$

where $\tilde{\Omega} = \tilde{T} + i\tilde{\Theta}$ is to be analytic on $\bar{\mathcal{D}} \setminus \{\pm i\}$. Note that (1.10) and (1.12) imply that

$$\tilde{\Omega}(\zeta) = \sum_{k=0}^{\infty} b_k \zeta^{2k}, \quad z \in \bar{\mathcal{D}} \setminus \{\pm i\} \quad ,$$

where all the b_k are real. The transformation (1.12) enables us to give a physical interpretation to $\tilde{\Omega}$ as follows:

$$\begin{aligned}(u - iv)(z) &= -\frac{d\tilde{w}}{dz}(z) = -\frac{dw}{d\zeta}(m(\zeta))\Big/\frac{dm}{d\zeta}(\zeta) = -c\{1 + \frac{i}{2}(1 + \zeta^2)f'(\zeta)\}^{-1} \\ &= -c\exp(-\tilde{T}(\zeta))\{\cos\tilde{\Theta}(\zeta) - i\sin\tilde{\Theta}(\zeta)\} \quad .\end{aligned} \tag{1.13}$$

Hence, $\tilde{\Theta}(\zeta) = \tilde{\Theta}(m^{-1}(z))$ is the angle between the negative velocity vector and the real axis in the z-plane, while $c\exp(-\tilde{T}(m^{-1}(z)))$ is the speed of the flow. Since (1.8a) is to be satisfied, it follows that $\Theta(t) = -\Theta(-t)$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, where $\Theta(t) = \tilde{\Theta}(e^{it})$ and $T(t) = \tilde{T}(e^{it})$. Since m takes $\{e^{it} : 0 < t < \frac{\pi}{2}\}$ onto

$\Gamma \cap \{x < 0\}$, and since $v < 0$ on $\Gamma \cap \{x < 0\}$, it follows that $0 < \Theta(t) < \frac{\pi}{2}$ on $(0,\frac{\pi}{2})$. The boundary condition (1.3) follows if

$$T(t) \to 0 \quad \text{as} \quad t \to \pm\frac{\pi}{2} , \tag{1.14a}$$

$$\Theta(t) \to 0 \quad \text{as} \quad t \to \pm\frac{\pi}{2} . \tag{1.14b}$$

With all this in mind, the free surface condition (1.7) now means that for $t \in (-\frac{\pi}{2},\frac{\pi}{2})$:

$$\begin{aligned} 0 &= -c^2\, T'(t)\exp(-2T(t)) + g\,\frac{dy}{dt} \\ &= -c^2\, T'(t)\exp(-2T(t)) + g\,\mathrm{Imag}(ie^{it}\,\frac{dm}{d\zeta}(e^{it})) \\ &= -c^2\, T'(t)\exp(-2T(t)) - \frac{2gh}{\pi}\sec t\,\sin\Theta(t)\exp(T(t)) \end{aligned} \tag{1.15}$$

by (1.11)-(1.13). Therefore

$$\frac{d}{dt}(\exp(-3T(t)) - \frac{6gh}{\pi c^2}\sec t\,\sin\Theta(t) = 0, \quad t \in (-\frac{\pi}{2},\frac{\pi}{2}) ,$$

and so if $\sec t \sin\Theta(t)$ is integrable on $(-\frac{\pi}{2},\frac{\pi}{2})$, then

$$\exp(-3T(t)) = \frac{q_c^3}{c^3} + \frac{6gh}{\pi c^2}\int_0^t \sec w\,\sin\Theta(w)\,dw ,$$

where $q_c = c\exp(-T(0))$ is the speed of the flow at the wave crest. Setting $\mu = \frac{6ghc}{\pi q_c^3}$ gives

$$-T(t) = \frac{1}{3}\{\ell n(1/\mu + \int_0^t \sec w\,\sin\Theta(w)dw) + \ell n(\frac{6gh}{\pi c^2})\}, \quad t \in (-\frac{\pi}{2},\frac{\pi}{2}) . \tag{1.16}$$

This last equation is an expression of the Bernoulli free surface condition in terms of the boundary value of the conjugate harmonic functions $\tilde{T}$ and $\tilde{\Theta}$ introduced in (1.12). The use of (1.16) with the fact that $\tilde{T}$ and $\tilde{\Theta}$ are conjugate harmonic

functions ensures that (cf. the proof of Theorem 1.1) Θ must satisfy

$$\Theta(s) = \frac{1}{3}\int_{-\pi/2}^{\pi/2}\left\{\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2ks \sin 2kt}{k}\right\}\frac{\sec t \sin\Theta(t)}{1/\mu+\int_0^t \sec w \sin\Theta(w)\,dw}\,dt, \quad s \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \quad .$$

If we can solve this equation for (μ,Θ), then the conjugate function T is given by (1.16), and so f' follows from (1.12). Hence, f may be determined, and then the mapping m is known, so that the complex potential $\tilde{w}(z) = w(m^{-1}(z))$ is known, and the problem is solved. The following theorem gives a rigorous proof of these statements.

THEOREM 1.1. Assume that Θ is an odd continuous function on $[-\frac{\pi}{2},\frac{\pi}{2}]$ with $0 < \Theta(s) \leq \pi$ on $(0,\frac{\pi}{2})$, $\Theta(\frac{\pi}{2}) = 0$, and $\sec t \sin\Theta(t)$ integrable on $(0,\pi)$. Suppose moreover that Θ satisfies the integral equation

$$\Theta(s) = \frac{1}{3}\int_{-\pi/2}^{\pi/2}\left\{\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2ks \sin 2kt}{k}\right\}\frac{\sec t \sin\Theta(t)}{1/\mu+\int_0^t \sec w \sin\Theta(w)\,dw}\,dt \tag{1.17}$$

for $s \in (-\frac{\pi}{2},\frac{\pi}{2})$ where $\mu > 0$. Then

(a) Θ is analytic on $(-\frac{\pi}{2},\frac{\pi}{2})$ and satisfies $0 < \Theta(s) < \frac{\pi}{2}$ on $(0,\frac{\pi}{2})$.

(b) Let $h,c,q_c > 0$ be any parameters satisfying the relations

$$\frac{6gh}{\pi c^2}\left(1/\mu + \int_0^{\pi/2} \sec w \sin\Theta(w)\,dw\right) = 1 \quad , \tag{1.18a}$$

$$\frac{\pi q_c^3}{6ghc} = \frac{1}{\mu} \quad , \tag{1.18b}$$

so that the square of the Froude number F is given by

$$F^2 = \frac{6}{\pi}\left(1/\mu + \int_0^{\pi/2} \sec w \sin\Theta(w)\,dw\right) \quad .$$

Then there exists a complex potential $\tilde{w}$ and a free surface Γ satisfying

(1.3)-(1.8). <u>The number</u> q_c <u>is the speed of the flow at the wave crest</u>.

<u>Proof</u>. (a) The first part of (a) is proved in section 4.3 and the second part in Theorem 3.7(b).

(b) Since $\sec t \sin \Theta(t)$ is integrable on $(0,\frac{\pi}{2})$, and $0 < \Theta < \pi$ there, it follows that

$$0 < \frac{\sec t \sin \Theta(t)}{1/\mu+\int_0^t \sec w \sin \Theta(w)dw} \in L_1(0,\frac{\pi}{2}) \quad .$$

Hence, by Fubini's Theorem and the regularity of Θ, it follows that for $s \in (-\frac{\pi}{2},\frac{\pi}{2})$,

$$\Theta(s) = \sum_{k=1}^{\infty} c_k \sin 2ks \quad ,$$

where

$$c_k = \frac{2}{3\pi k}\int_0^{\pi/2} \frac{\sin 2kt \sin \Theta(t) \sec t}{1/\mu+\int_0^t \sec w \sin \Theta(w)dw} dt$$

$$= \frac{-4}{3\pi}\int_0^{\pi/2} \cos 2kt \,\ell n(1/\mu + \int_0^t \sec w \sin \Theta(w)dw)dt$$

by an integration by parts. It follows at once that for $s \in (-\frac{\pi}{2},\frac{\pi}{2})$,

$$\ell n(1/\mu + \int_0^s \sec w \sin \Theta(w)dw) = c_0 - 3\sum_{k=1}^{\infty} c_k \cos 2ks \quad , \tag{1.19}$$

where

$$c_0 = \frac{2}{\pi}\int_0^{\pi/2} \ell n(1/\mu + \int_0^t \sec w \sin \Theta(w)dw)dt \quad .$$

Define $\tilde{\Omega} = \tilde{T} + i\tilde{\Theta}$ by

$$\tilde{\Omega}(\zeta) = -\frac{1}{3}\ell n(\frac{6gh}{\pi c^2}) - \frac{c_0}{3} + \sum_{k=1}^{\infty} c_k \zeta^{2k}, \quad \zeta \in \mathcal{D} \quad , \tag{1.20}$$

c and h are chosen to satisfy (1.18a) and the coefficients c_k are all

real. The analyticity of $\Theta(t) = \tilde{\Theta}(e^{it})$ on $(-\frac{\pi}{2},\frac{\pi}{2})$ ensures that $\tilde{\Omega}$ is analytic on $\bar{\mathcal{D}} \setminus \{\pm i\}$. The use of (1.19) yields

$$\begin{aligned} T(t) = \tilde{T}(e^{it}) &= -\frac{1}{3}\{\ell n(1/\mu + \int_0^t \sec w \sin \Theta(w)dw) + \ell n(\frac{6gh}{\pi c^2})\} \\ &= -\frac{1}{3}\{\ell n(1/\mu + \int_0^t \sec w \sin \Theta(w)dw) - \ell n(1/\mu + \int_0^{\pi/2} \sec w \sin \Theta(w)dw)\}, \end{aligned} \tag{1.21}$$

and note that $\tilde{T}$ is continuous on $\partial\mathcal{D}$. The hypothesis of the theorem ensures that $\tilde{\Theta}$ is continuous on $\partial\mathcal{D}$, and harmonic analysis then shows that $\tilde{\Omega}$ is continuous on $\partial\mathcal{D}$. Equation (1.21) and the hypothesis $\Theta(\pm\frac{\pi}{2}) = 0$ together give

$$\tilde{\Theta}(\zeta),\ \tilde{T}(\zeta) \to 0 \quad \text{as} \quad \zeta \in \bar{\mathcal{D}} \to \pm i \ . \tag{1.22}$$

Define the function f from the representation

$$1 + \frac{i}{2}(1 + \zeta^2)f'(\zeta) = \exp(\tilde{\Omega}(\zeta))$$

with the normalization $f(0) = 0$; note that f is analytic on $\bar{\mathcal{D}} \setminus \{\pm i\}$ since $\tilde{\Omega}$ is analytic there. It follows from (1.20) that f has the form

$$f(\zeta) = \sum_{k=0}^{\infty} a_k (i\zeta)^{2k+1}, \quad \zeta \in \bar{\mathcal{D}} \setminus \{\pm i\} \ ,$$

where the coefficients a_k are all real. From this representation, we have

$$\overline{f(\zeta)} = f(-\bar{\zeta}) = -f(\bar{\zeta}) \quad \text{for } \zeta \in \bar{\mathcal{D}} \setminus \{\pm i\} \ . \tag{1.23}$$

A calculation yields

$$\frac{d}{dt} f(e^{it}) = \sec t\{\exp(\Omega(t))-1\} \ ,$$

and so

$$\frac{d}{dt} \text{Imag } f(e^{it}) = \sec t \sin \Theta(t) \exp(T(t)) \in L_1(-\frac{\pi}{2},\frac{\pi}{2})$$

by hypothesis. It follows that the even function Imag $f(e^{it})$ has a limit as $t \to \pm\frac{\pi}{2}$, and (1.23) ensures that this limit must be zero. Hence, Imag f is

continuous on $\partial\mathcal{D}$, and so harmonic analysis shows that Imag f is continuous on $\bar{\mathcal{D}}$; in particular,

$$\lim_{\zeta\in\bar{\mathcal{D}}\to\pm i} \text{Imag } f(\zeta) = 0 \quad . \tag{1.24}$$

If we define the analytic function m by

$$m(\zeta) = -\frac{2h}{\pi}\{\ln(\frac{i+\zeta}{i-\zeta}) + f(\zeta)\}, \quad \zeta \in \bar{\mathcal{D}} \setminus \{\pm i\} \quad ,$$

then the function m is analytic on $\bar{\mathcal{D}} \setminus \{\pm i\}$ since f is analytic there, and m satisfies $\overline{m(\zeta)} = m(-\bar{\zeta}) = -m(\bar{\zeta})$ by (1.23). Furthermore,

$$\frac{dm}{d\zeta}(\zeta) = \frac{4hi}{\pi(1+\zeta^2)} \exp(\tilde{\Omega}(\zeta)) \neq 0 \quad \text{in} \quad \bar{\mathcal{D}} \setminus \{\pm i\} \quad ,$$

and

$$\frac{dm}{dt}(e^{it}) = -\frac{2h}{\pi} \sec t \exp(\Omega(t)), \quad t \in (-\frac{\pi}{2},\frac{\pi}{2}) \quad .$$

Since $\Theta(t) \in (-\frac{\pi}{2},\frac{\pi}{2})$ for $t \in [-\frac{\pi}{2},\frac{\pi}{2}]$, it follows that

$$\frac{d}{dt} \text{Real } m(e^{it}) = -\frac{2h}{\pi} \sec t \exp(T(t))\cos\Theta(t) \leq -\text{ const. } \sec t \quad , \tag{1.25a}$$

where the constant is positive and independent of t. Hence, the map $t \mapsto m(e^{it})$ is injective for $t \in (-\frac{\pi}{2},\frac{\pi}{2})$. Furthermore, the use of (1.21) in (1.25a) implies that

$$\text{Real } m(e^{it}) = -\frac{2h}{\pi}(1/\mu + \int_0^{\pi/2} \sec w \sin\Theta(w)dw)^{1/3} \int_0^t \sec s \cos\Theta(s)(1/\mu + \int_0^s \sec w \sin\Theta(w)dw)^{-1/3}ds \; , \tag{1.25b}$$

so that $m(\pm\frac{\pi}{2}) = \mp\infty$. A similar result holds for $t \in (\frac{\pi}{2},\frac{3\pi}{2})$, and a variant of the argument principle then ensures that m is injective in $\bar{\mathcal{D}} \setminus \{\pm i\}$. Hence, m^{-1} exists and is analytic on the closure of $\{m(\zeta) : \zeta \in \mathcal{D}\}$.

To complete the proof of (b), we set $\tilde{w}(z) = w(m^{-1}(z))$, where

$$w(z) = \frac{-2ch}{\pi} \ln(\frac{i+z}{i-z}) \quad ,$$

and we now check that (1.3)-(1.8) are satisfied. The calculation for (1.13) yields

$$\begin{aligned}(u - iv)(z) &= -c \exp(-\tilde{\Omega}(m^{-1}(z))) \\ &= -c \exp(-\tilde{T}(m^{-1}(z)))\{\cos \tilde{\Theta}(m^{-1}(z)) - i \sin \tilde{\Theta}(m^{-1}(z))\} \to -c \quad \text{as} \quad |z| \to \infty\end{aligned}$$

by (1.22), and so (1.3) is satisfied. Next, we recall that the solitary wave Γ is given by

$$\Gamma = \{m(e^{it}) : t \in (-\frac{\pi}{2},\frac{\pi}{2})\} \quad ,$$

and the use of (1.24) immediately gives (1.4). If $z \in \Gamma$, then $m^{-1}(z) = e^{it}$ for some $t \in (-\frac{\pi}{2},\frac{\pi}{2})$, whence

$$\tilde{\psi}(z) = \text{Imag } w(z) = \text{Imag}\{\frac{-2ch}{\pi} \ln(\frac{i+e^{it}}{i-e^{it}})\} = ch \quad ,$$

and so (1.5) holds. If $\text{Imag } z = 0$, then $m^{-1}(z) = i\eta$ for some $\eta \in (-1,1)$, and then

$$\tilde{\psi}(z) = \text{Imag}\{\frac{-2ch}{\pi} \ln(\frac{i+i\eta}{i-i\eta})\} = 0 \quad ,$$

whence (1.6) holds. Equation (1.7) follows immediately from (1.21) and the arguments for (1.15). The symmetry of Γ about the y-axis, condition (1.8a), follows from the relation $m(\bar{\zeta}) = -\overline{m(\zeta)}$. Now as $t \to -\frac{\pi}{2}$, equation (1.25b) gives $m(e^{it}) \to \infty$, and so

$$\frac{d}{dt} \text{Imag } m(e^{it}) = \frac{-2h}{\pi} \sec t \exp(T(t)) \sin \Theta(t) > 0$$

since $0 > \Theta(t) > -\frac{\pi}{2}$ on $(-\frac{\pi}{2},0)$. Hence, the function H is decreasing for $x > 0$, and so (1.8b) is satisfied. q.e.d.

It is shown in section 4.5 that if the hypothesis of Theorem 1.1 is satisfied, then $\Theta(s) \leq \text{const.}(\frac{\pi}{2} - s)^{\alpha}$ for $s \in (0,\frac{\pi}{2})$ and some $\alpha > 0$. It then follows that

$$\frac{d}{dt} f(e^{it}) = \sec t\{\exp(\Omega(t)) - 1\} \in L_1(-\frac{\pi}{2},\frac{\pi}{2}) \quad .$$

Hence, f is continuous on $\partial\mathcal{D}$, and so f is continuous on $\bar{\mathcal{D}}$; this result adds to that of (1.24).

For convenience with notation, we make a change of variables in (1.17). Put

$$\left.\begin{aligned} \theta(t) &= \Theta(\tfrac{t}{2}) \quad , \\ f(t) &= \tfrac{1}{2} \sec \tfrac{t}{2} \; , \end{aligned}\right\} \; t \in (-\pi,\pi) \quad ,$$

so that θ is to satisfy the equation

$$\theta(s) = \frac{2}{3} \int_0^{\pi} G(s,t) \frac{f(t)\sin\theta(t)}{1/\mu + \int_0^t f(w)\sin\theta(w)dw} dt \quad \text{for} \quad s \in (0,\pi), \tag{1.26}$$

where

$$G(s,t) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k} \quad .$$

Theorem 1.1 shows that the solitary wave problem will be solved if we can prove the existence of a function $\theta \in C[0,\pi]$ which satisfies (1.26) and for which $\theta(s) \leq \pi$ on $(0,\pi)$, $\theta(0) = \theta(\pi) = 0$, and $f \sin\theta \in L_1(0,\pi)$.

1.3 Methods and results

As in the periodic case, the key to a successful existence theory in the large for solitary water waves lies with the theory of nonlinear positive operators in Banach spaces. The basic existence theory for periodic water waves follows from an application of this general theory to Nekrassov's integral equation; for solitary waves, it does not. The Appendix contains basic definitions and results for positive operator theory.

Both Nekrassov integral equations, (1.1) for the periodic problem and (1.26) for the solitary wave, may be written in the form

$$\theta = A(\mu,\theta) \quad , \tag{1.27}$$

and θ, the angle between the free surface and the horizontal, is required to be non-negative on $[0,\pi]$. While this suggests the use of positive operator theory, care must be exercised because (1.27) is not a positive operator equation even though Theorem 2.5 (a) shows that G is a non-negative kernel. Define a continuous function $J : \mathbb{R} \to \mathbb{R}$ as follows:

$$Jx = \begin{cases} x & \text{if } |x| \le \pi \quad , \\ \pi & \text{if } x \ge \pi \quad , \\ -\pi & \text{if } x \le -\pi \quad . \end{cases} \tag{1.28}$$

In the periodic problem, for the case of waves on an infinitely deep flow, we put

$$A(\mu,\theta) = \frac{2}{3}\int_0^\pi G(s,t) \frac{\dfrac{\sin(J\theta(t))}{t}}{1/\mu + \int_0^t \sin(J\theta(w))dw}\, dt \quad .$$

Then (1.27) is a completely continuous operator equation with respect to K_0, the cone of non-negative functions in $C_0[0,\pi]$, and is of the type to which Theorem A4 applies. Therefore, there exists an unbounded, closed, connected set $\mathcal{C}$ of solutions of (1.27) in $(0,\infty) \times K_0$ such that $(3,0) \in \mathcal{C}$ and $\theta \neq 0$ if $(\mu,\theta) \in \mathcal{C}\backslash\{(3,0)\}$. (The number 3 is the smallest characteristic value* of the linearized problem $\theta(s) = \frac{2}{3}\int_0^\pi G(s,t)\theta(t)dt$, it is simple, and the corresponding eigenfunction $\sin s$ lies in K_0.) Subsequent analysis of (1.27) establishes the a priori bounds $0 < \theta(s) < \frac{\pi}{2}$ for $s \in (0,\pi)$, and $\mu > 3$ if $(\mu,\theta) \in \mathcal{C}\backslash\{(3,0)\}$.

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*A characteristic value of an operator is the reciprocal of an eigenvalue.

For two fundamental reasons, a similar approach to the solitary wave problem fails. If we put

$$A(\mu,\theta) = \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f(t)\sin(J\theta(t))}{1/\mu+\int_0^t f(w)\sin(J\theta(w))dw}\,dt \quad , \tag{1.29}$$

then, formally, A is a positive operator. Furthermore, A is a continuous map from $(0,\infty)\times K_0$ into K_0; unfortunately, the singularity in f at π prevents this map from being completely continuous. Another difficulty follows from the fact that the solitary wave problem linearized about $\theta = 0$,

$$\theta(s) = \frac{2\mu}{3}\int_0^\pi G(s,t)f(t)\theta(t)dt \quad ,$$

has no eigenfunction in K_0. Thus the local bifurcation problem is not standard since the linearized problem has no solution.

These difficulties can be overcome by studying a sequence of non-singular problems:

$$\theta^{(n)}(s) = \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f_n(t)\sin(J\theta^{(n)}(t))}{1/\mu+\int_0^t f_n(w)\sin(J\theta^{(n)}(w))dw}\,dt \quad , \tag{1.30}$$

where for each positive integer n,

$$f_n(t) = \begin{cases} f(t) & \text{if } |t| \le \pi - \frac{1}{n} \,, \\ f(\pi - \frac{1}{n}) & \text{if } \pi - \frac{1}{n} \le t \le \pi \,. \end{cases}$$

It is an easy consequence of the linear theory of positive operators summarized in the appendix that the corresponding linear characteristic value problem

$$\phi(s) = \frac{2\mu}{3}\int_0^\pi G(s,t)f_n(t)\phi(t)dt \tag{1.31}$$

has a unique normalized solution $(\lambda_n,\phi_n) \in (0,\infty)\times K_0$. From a variational

characterisation of λ_n, it follows that $\lambda_n \downarrow \frac{6}{\pi}$ as $n \to \infty$ (Theorem 3.1). The global bifurcation result (Theorem A4) applies at once to (1.30), and after further detailed analysis, we arrive at the following results (Theorem 3.5):

(i) for each positive integer n, there exists an unbounded, closed, connected set $C_n \subset (0,\infty) \times K_0$ of solutions $(\mu^{(n)},\theta^{(n)})$ of (1.30),

(ii) $(\lambda_n,0) \in C_n$ and $\theta^{(n)} \neq 0$ if $(\mu^{(n)},\theta^{(n)}) \in C_n\backslash\{(\lambda_n,0)\}$,

(iii) $\{\mu^{(n)}: (\mu^{(n)},\theta^{(n)}) \in C_n\} = [\lambda_n,\infty)$.

(iv) If $(\mu^{(n)},\theta^{(n)}) \in C_n\backslash\{(\lambda_n,0)\}$, then $0 < \theta^{(n)}(s) < \pi$ on $(0,\pi)$ and $\theta^{(n)}(s) < \frac{\pi}{2}$ on $[0,\pi - \frac{1}{n}]$.

In section 3.3, we show that the sets C_n converge in a certain sense to a set C and that

(a) C is an unbounded, closed, connected set in $(0,\infty) \times K_0$ of solutions (μ,θ) of (1.26),

(b) $(\frac{6}{\pi},0) \in C$, and $\theta \neq 0$ if $(\mu,\theta) \in C\backslash\{(\frac{6}{\pi},0)\}$,

(c) $\{\mu : (\mu,\theta) \in C\} = [\frac{6}{\pi},\infty)$.

(d) If $(\mu,\theta) \in C\backslash\{(\frac{6}{\pi},0)\}$, then $0 < \theta(s) < \frac{\pi}{2}$ on $(0,\pi)$.

We show in section 4.3 that θ is analytic on $(-\pi,\pi)$, and so if $(\mu,\theta) \in C\backslash\{(\frac{6}{\pi},0)\}$, then (μ,Θ), where $\Theta(s) = \theta(2s)$, satisfies the hypotheses of Theorem 1.1. Further properties of C are given in sections 3 and 4, and the most important of these are now presented.

(e) If $\beta \in (0,\frac{\pi}{6})$, then there exists $(\mu,\theta) \in C$ such that $\max_{s\in[0,\pi]} \theta(s) = \beta$.

(f) For any flow parameters c and h satisfying the relation

$$\frac{c^2}{gh} = F^2 = \frac{6}{\pi}(1/\mu + \int_0^\pi f(t)\sin\theta(t)dt) ,$$

there exist solitary waves $\Gamma = \{(x,y) : y = H(x),\ x \in \mathbb{R}\}$, whose amplitudes are given by

$$H(0) - H(-\infty) = H(0) - h = \frac{h}{2}\{F^2 - (\frac{6F}{\pi\mu})^{2/3}\} .$$

The exact shape of the free surface, given by H, may be calculated explicitly as in (4.3).

(g) The Froude number F satisfies $F \geq 1$ for any $(\mu,\theta) \in \mathcal{C}$.

In section 4.5, we give various properties of F and show that

(h) if $(\mu,\theta) \in \mathcal{C}$ and $F > 1$, then $(\pi-t)^{-\alpha}\theta(t) \to 0$ as $t \to \pi$ for all $\alpha \in [0,\tilde{\alpha})$, where $\tilde{\alpha}$ is the unique number such that

$$F^2 = \frac{2}{\pi\tilde{\alpha}} \tan(\frac{\pi\tilde{\alpha}}{2}) .$$

This result translates to one for the solitary wave profile Γ:

$$|H(x) - H(-\infty)| = |H(x) - h| \leq \text{const.} \exp(\frac{-\pi\alpha|x|}{2h}), \quad x \in \mathbb{R} .$$

It is generally accepted that solitary waves only propagate on flows with $F > 1$. Such demonstrations of this fact as there are in the literature [30], [42] depend on a certain integral, the mass, being finite. We show in Theorem 4.8 that for non-trivial solutions (μ,θ) in $\mathcal{C}$, the mass is finite if and only if $F > 1$. We prove in Theorem 4.12 that

(i) if $(\mu,\theta) \in \mathcal{C}\backslash\{(\frac{6}{\pi},0)\}$, then $F > 1$.

The last section of this paper is devoted to a discussion of the existence and properties of a solution θ of (1.26) in the limiting case $\frac{1}{\mu} = 0$. The function has the property $0 < \theta(s) < \frac{\pi}{2}$ on $(0,\pi)$ and satisfies the equation

$$\theta(s) = \frac{2}{3}\int_0^\pi G(s,t) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt \qquad (1.32)$$

on $(0,\pi)$. Equation (1.32) is the Nekrasov integral equation for solitary waves with a stagnation point at the wave crest $(1/\mu = \frac{\pi q_c^3}{6ghc} = 0)$, the so-called <u>solitary wave of greatest height</u>. Stokes' conjecture is that this wave is not smooth at its

crest, and that the free surface on either side of the crest subtends an angle of $\frac{\pi}{6}$ with the horizontal, that is, $\lim_{s \to 0+} \theta(s) = \frac{\pi}{6}$. We are unable to prove this result, although all the evidence seems to indicate its veracity. For example, if $\lim_{s \to 0+} \theta(s)$ exists, then it is easy to see that it must be $\frac{\pi}{6}$. Indeed, we shall prove that an even weaker regularity result leads to a verification of Stokes' conjecture: the following are equivalent,

(i) $\lim_{s \to 0+} \theta(s) \neq \frac{\pi}{6}$;

(ii) $\liminf_{s \to 0+} s^{-1} \int_0^s \sin \theta(t) dt < \frac{1}{2} < \limsup_{s \to 0+} s^{-1} \int_0^s \sin \theta(t) dt$.

This equivalence is a consequence of (1.32) and a detailed study of θ near 0. So far we have only been able to prove that there exists a number $a > 0$ such that

$$\liminf_{s \to 0+} \theta(s) = a \quad . \tag{1.33}$$

As a consequence of (1.33) and the work in section 4, it follows that for the solitary wave of greatest height,

$$F^2 = \frac{6}{\pi} \int_0^\pi f(t) \sin \theta(t) dt > 1 \quad ,$$

and that

$$H(0) - H(-\infty) = H(0) - h = \frac{h}{2} F^2 = \frac{1}{2} \frac{c^2}{g} \quad ,$$

as is to be expected in this case. The wave profile may be determined as in (4.3) with $\mu = \infty$, and its decay at infinity is related to the Froude number as before.

2. PRELIMINARY RESULTS AND FUNCTION SPACES

Throughout this paper, functions are assumed to take values in either $\mathbb{R}$ or $\mathbb{C}$. If $U \subset \mathbb{R}^n$ is a Lebesgue measurable set, let $L_p(U)$, $p \geq 1$, denote the Banach space of p-th power integrable 'functions' on U; the norm is given by

$$|f|_{L_p(U)} = \left(\int_U |f|^p\right)^{1/p} .$$

If U is an open set and $f : U \to \mathbb{R}$ (or $\mathbb{C}$), we write $f \in C^k(U)$ if f and all of its partial derivatives of order up to and including k are continuous in U. The Banach space $C^k(\bar{U})$ denotes those functions in $C^k(U)$ whose derivatives up to and including k have continuous extension onto $\bar{U} = U \cup \partial U$, where ∂U denotes the boundary of U; the norm is

$$|f|_{C^k(\bar{U})} = \sup_{x \in U, 0 \leq |\beta| \leq k} |D^\beta f(x)| ,$$

where $\beta = (\beta_1, \ldots, \beta_n)$, $D^\beta f = D_1^{\beta_1} \ldots D_n^{\beta_n}$ and $|\beta| = \beta_1 + \ldots + \beta_n$. We write $C_0(\bar{U}) = \{f \in C(\bar{U}) : f = 0 \text{ on } \partial U\}$. If $f \in C(\bar{U})$ and

$$|f|_{C^\alpha(\bar{U})} = \sup_{x,y \in U, |x-y|<1} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$$

for some $\alpha \in (0,1)$, then f is said to be Hölder continous with exponent α on $\bar{U}$, and we write $f \in C^\alpha(\bar{U})$. The Banach space $C^{k,\alpha}(\bar{U})$ consists of those functions in $C^k(\bar{U})$ for which $D^\beta f \in C^\alpha(\bar{U})$, $|\beta| = k$; the norm is given by

$$|f|_{C^{k,\alpha}(\bar{U})} = |f|_{C^k(\bar{U})} + \max_{|\beta|=k} |D^\beta f|_{C^\alpha(\bar{U})} .$$

For convenience, we write $C^{k,\alpha}[a,b]$ for $C^{k,\alpha}([a,b])$, $C^k(a,b)$ for $C^k((a,b))$, and $C_0[a,b]$ for $C_0([a,b])$ where $[a,b] \subset \mathbb{R}$. We write $C[a,b)$ and $C(a,b]$ for those functions in $C(a,b)$ which have limits at a and b, respectively.

The following result is standard and a proof may be found in [17].

THEOREM 2.1. Let B denote the open unit ball in $\mathbb{R}^2$ and let $\phi \in C^{1,\alpha}[-\pi,\pi]$ for some $\alpha \in (0,1)$. If $u \in C(\bar{B}) \cap C^2(B)$ and satisfies

$$\Delta u = 0 \quad \text{in} \quad B \quad ,$$

$$u(\cos t, \sin t) = \phi(t), \; t \in [-\pi,\pi] \quad ,$$

then $u \in C^{1,\alpha}(\bar{B})$.

In the following material up to and including Theorem 2.4, all functions f are defined on $\mathbb{R}$ and are 2π-periodic. A trigonometric series

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos ks + b_k \sin ks) \tag{2.1}$$

is said to be the Fourier series of a function $f \in L_1(-\pi,\pi)$ if

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \quad ,$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos kx\, dx, \; b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin kx\, dx, \; k = 1,2,\ldots \quad .$$

We write this as

$$f \sim a_0 + \sum_{k=1}^{\infty} (a_k \cos ks + b_k \sin ks) \quad . \tag{2.2}$$

The trigonometric series

$$\sum_{k=1}^{\infty} (a_k \sin ks - b_k \cos ks) \tag{2.3}$$

is called the conjugate of the series (2.2). It does not necessarily follow that (2.3) is a Fourier series, but if it is, and if (2.2) is the Fourier series of f, then we write

$$\underset{\sim}{C}f \sim \sum_{k=1}^{\infty} (a_k \sin ks - b_k \cos ks) \quad .$$

The following results on trigonometric series are used frequently in this paper. If $f \in L_1(-\pi,\pi)$, then we define $\hat{f}(x)$, when it exists, as follows:

$$\hat{f}(x) = \lim_{\varepsilon\to 0} -\frac{1}{\pi}\int_{\varepsilon}^{\pi} \frac{f(x+t) - f(x-t)}{2\tan\frac{t}{2}}\,dt \quad ;$$

it is known [1; II, p. 52] that this limit exists almost everywhere.

THEOREM 2.2. (a) *If* $f \in C^{0,\alpha}[-\pi,\pi]$ *for some* $\alpha \in (0,1)$, *then* $\hat{f} \in C^{0,\alpha}[-\pi,\pi]$.

(b) *Assume that* $f \in L_1(-\pi,\pi)$ *and* $f \in C^{0,\alpha}[a,b]$ *for some non-trivial closed interval* $[a,b] \subset (-\pi,\pi)$. *Then* $\hat{f} \in C^{0,\alpha}[a',b']$, *where* $a < a' < b' < b$, *and* (2.2) *and* (2.3) *converge uniformly on* $[a',b']$ *to* f *and* $\hat{f}$, *respectively. Moreover*

$$|\hat{f}|_{C^{0,\alpha}[a',b']} \le \text{const.}\left(|f|_{L_1(-\pi,\pi)} + |f|_{C^{0,\alpha}[a,b]}\right) \quad ,$$

where the constant depends on a, a', b, b', *and* α, *but not on* f.

Part (a) of Theorem 2.2 is Privalov's theorem [1; II, p. 99], (b) is a simple variant of it, and the uniform convergence is standard [1; I, p. 320, II, p. 136]. The following theorem is due to M. Riesz [1; II, p. 106].

THEOREM 2.3. *If* $f \in L_p(-\pi,\pi)$, $p > 1$, *then* $\hat{f} \in L_p(-\pi,\pi)$, *and there exists a constant*, A_p , *independent of* f, *such that*

$$\int_{-\pi}^{\pi} |\hat{f}(x)|^p dx \le A_p \int_{-\pi}^{\pi} |f(x)|^p dx \quad .$$

Moreover, $\hat{f} = \underset{\sim}{C}f$; *that is, the Fourier series of* $\hat{f}$ *is the conjugate of that for* f.

The following result is standard, and a proof may be found in [1; I, p. 150].

THEOREM 2.4. *If* (2.2) *is the Fourier series of* $f \in L_1(-\pi,\pi)$ *and if* $f(x_0 \pm 0)$ *both exist at* $x_0 \in [-\pi,\pi]$, *then*

$$\lim_{r\to 1} \{a_0 + \sum_{k=1}^{\infty} r^k(a_k \cos kx_0 + b_k \sin kx_0)\} = \frac{1}{2}(f(x_0 + 0) + f(x_0 - 0)) \quad .$$

Finally, we will need the following results about the kernel of the operator in (1.26). For $(s,t) \in [-\pi,\pi] \times [-\pi,\pi]$, $s \neq t$, define

$$G(s,t) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k} .$$

THEOREM 2.5. (a) $G(s,t) = \frac{1}{4\pi} \ln\{\frac{1-\cos(s+t)}{1-\cos(s-t)}\} \geq 0$, for all $(s,t) \in [0,\pi] \times [0,\pi] \setminus \{(t,t) : t \in [0,\pi]\}$.

(b) For each $p > 1$, the map $u \mapsto \int_0^\pi G(\cdot,t)u(t)dt$ is completely continuous from $L_p(0,\pi)$ into $L_p(0,\pi)$, from $L_p(0,\pi)$ into $C[0,\pi]$, and from $C[0,\pi]$ into itself.

(c) For each interval $[a,b] \subset (0,\pi)$, there exists $\delta > 0$ such that

$$G(s,t) \geq \delta \sin s$$

for all $(s,t) \in [0,\pi] \times [a,b]$.

(d) Let u be a function defined on $(0,\pi)$ and assume that $G(s,\cdot)u(\cdot) \in L_1(0,\pi)$ for all $s \in (0,\pi)$. If

$$v(s) = \int_0^\pi G(s,t)u(t)dt ,$$

then

$$v(\pi-s) = \int_0^\pi G(s,t)u(\pi-t)dt .$$

(e) $\frac{2}{3} \int_0^\pi G(s,t)\frac{1}{2} \tan \frac{t}{2}\, dt = \frac{s}{6}$ for all $s \in (0,\pi)$.

(f) $G(s,t) = \frac{1}{2\pi} \ln\left|\frac{2\pi-s-t}{s-t}\right| + h(s,t) = \frac{1}{2\pi} \ln\left|\frac{s+t}{s-t}\right| + \tilde{h}(s,t)$, where h and $\tilde{h}$ are analytic on $\mathbb{R}^2$, and there exists a constant A such that

$$|h(s,t)| \leq A|\pi-s|\,|\pi-t| , \quad |\tilde{h}(s,t)| \leq A|st| \quad \forall (s,t) \in \mathbb{R}^2 .$$

(g) If $u \in L_1(-\pi,\pi)$, then

$$\int_{-\pi}^{\pi} G(s,\tau)\{\int_{-\pi}^{\pi} G(\tau,t)u(t)dt\}d\tau = \int_{-\pi}^{\pi} G_1(s,t)u(t)dt \quad ,$$

<u>where</u>

$$G_1(s,t) = \frac{1}{2\pi}\begin{cases} t(\pi-s) & \text{for } t \le s \ , \\ s(\pi-t) & \text{for } t \ge s \ . \end{cases}$$

(h) <u>Let</u> $\delta \in (0,\frac{\pi}{2})$. <u>If</u> $u(s) = \int_{\delta}^{\pi-\delta} G(s,t)v(t)dt$ <u>for</u> $s \in (0,\pi)$, <u>where</u> $v \in C^{\alpha}[\delta,\pi-\delta]$, <u>then</u> $u \in C^{1,\alpha}[a,b]$ <u>for all closed intervals</u> $[a,b] \subset (\delta,\pi-\delta)$.

<u>Proof</u>. The proofs of (a) and (b) appear in [19].

(c) Part (a) implies that

$$\frac{\partial G}{\partial s} = \frac{1}{2\pi}\frac{\sin t}{(\cos s - \cos t)}$$

and

$$\frac{\partial^2 G}{\partial s^2} = \frac{1}{2\pi}\frac{\sin s \sin t}{(\cos s - \cos t)^2} \ge 0$$

for $(s,t) \in [0,\pi] \times [0,\pi]$, $s \ne t$. Since

$$\left.\frac{\partial G}{\partial s}\right|_{s=0} = \frac{1}{2\pi}\frac{\sin t}{(1-\cos t)}$$

and

$$\left.\frac{\partial G}{\partial s}\right|_{s=\pi} = -\frac{1}{2\pi}\frac{\sin t}{(1+\cos t)} \quad ,$$

it follows that

$$G(s,t) \ge \begin{cases} \frac{1}{2\pi}(\frac{\sin t}{1-\cos t})\sin s & , \quad s \in [0,t) \ , \\ \frac{1}{2\pi}(\frac{\sin t}{1+\cos t})\sin s & , \quad s \in [t,\pi] \ , \end{cases}$$

and the result follows immediately.

(d) Part (a) gives $G(\pi-s, \pi-t) = G(s,t)$, and so

$$v(\pi-s) = \frac{2}{3}\int_0^{\pi} G(\pi-s,t)u(t)dt = \frac{2}{3}\int_0^{\pi} G(\pi-s,\pi-t)u(\pi-t)dt = \frac{2}{3}\int_0^{\pi} G(s,t)u(\pi-t)dt \ .$$

(e) Part (f) ensures that $G(s,t) = O(\pi-t)$ for each $s \in (0,\pi)$, and so $G(s,t)\tan\frac{t}{2} \in L_1(0,\pi)$ as a function of t. The use of (d) yields

$$\chi(\pi-s) = \frac{2}{3}\int_0^\pi G(s,t)\frac{1}{2}\cot\frac{t}{2}\,dt \quad ,$$

where

$$\chi(s) = \frac{2}{3}\int_0^\pi G(s,t)\frac{1}{2}\tan\frac{t}{2}\,dt \quad .$$

Let η be the odd extension of $\chi(\pi-s)$ to $(-\pi,\pi)$ so that $\eta \sim \sum_{k=1}^{\infty} a_k \sin ks$, where

$$a_k = \frac{1}{3k}\int_0^\pi \sin kt\,\frac{1}{2}\cot\frac{t}{2}\,dt = \frac{1}{3k} \quad .$$

Hence, $\eta \sim \sum_{k=1}^{\infty} \frac{\sin ks}{3k} = \frac{\pi-s}{6}$ on $(0,\pi)$. Since $\eta \in C^{0,\alpha}[a,b]$ for all $[a,b] \subset (0,\pi)$ and all $\alpha \in (0,1)$, Theorem 2.2(b) ensures that $\chi(\pi-s) = \frac{\pi-s}{6}$ on compact subsets of $(0,\pi)$.

(f) If $(s,t) \in [0,\pi] \times [0,\pi]$, $s \neq t$, then (a) gives

$$h(s,t) = G(s,t) - \frac{1}{2\pi}\ln\left|\frac{2\pi-s-t}{s-t}\right| = \frac{1}{4\pi}\ln\left\{\frac{(1-\cos(2\pi-s-t))(s-t)^2}{(1-\cos(s-t))(2\pi-s-t)^2}\right\}$$

$$= \frac{1}{4\pi}\ln\left(1 + \frac{w(2\pi-s-t) - w(s-t)}{w(s-t)}\right) \quad ,$$

where $w(x) = x^{-2}(1-\cos x)$. Since $w(s-t) \geq \frac{2}{\pi^2}$ for $s,\, t \in [0,\pi]$, we have

$$|h(s,t)| \leq \frac{\pi}{8}\,|w(2\pi-s-t)-w(s-t)| = \frac{\pi}{8}\,|w(\pi-s+\pi-t)-w(\pi-t+s-\pi)| \leq \text{const.}\,|\pi-s|\;|\pi-t|$$

by examining the Taylor series for w. Finally, since $G(s,t) = G(\pi-s,\pi-t)$, the result for $\tilde{h}$ follows.

(g) This follows immediately from the definition of G and the fact [18; p. 263, (31.1.8)] that

$$\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k^2} = G_1(s,t) \quad .$$

(h) Identify v with its extension to $[0,\pi]$ as zero outside $[\delta, \pi-\delta]$. Now identify u and v with their odd extensions to $[-\pi,\pi]$. Then

$$u(s) = \frac{1}{2} \int_{-\pi}^{\pi} G(s,t)v(t)dt \quad ,$$

and it follows by Fubini's theorem that

$$u' = - \frac{1}{2} \underset{\sim}{C}v \quad .$$

The result follows from Theorem 2.2(b). q.e.d.

3. EXISTENCE THEORY FOR FINITE-AMPLITUDE SOLITARY WAVES

3.1. The linearized approximate problem

We have summarized, in the Appendix, various definitions and results from the spectral theory of positive, linear operators on cones in Banach spaces [22], which are used in this section. Also recorded there are results about the global bifurcation of positive solutions of nonlinear eigenvalue problems [14], [48], which are called upon in the next section.

The linearization of (1.30) about the trivial solution is

$$\varphi = \mu L_n \varphi \quad , \tag{3.1}$$

where

$$L_n \varphi(s) = \frac{2}{3} \int_0^\pi G(s,t) f_n(t) \varphi(t) dt \quad , \tag{3.2}$$

and f_n is defined as in (1.30). If K_0 denotes the reproducing cone of non-negative functions in $C_0[0,\pi]$, then L_n is a completely continuous linear operator from $C_0[0,\pi]$ into itself which leaves K_0 invariant.

THEOREM 3.1. (a) For each $n \geq 1$, there exists a unique solution (λ_n, φ_n) of the characteristic value problem

$$(\lambda, \varphi) \in [0,\infty) \times K_0, \quad |\varphi|_{C_0[0,\pi]} = 1, \quad \varphi = \lambda L_n \varphi \quad . \tag{3.3}$$

Moreover λ_n^{-1} is the spectral radius of $L_n : C_0[0,\pi] \to C_0[0,\pi]$.

(b) For each $n \geq 1$,

$$\lambda_{n+1} \leq \lambda_n \quad \text{and} \quad \lambda_n \to \frac{6}{\pi} \quad \text{as} \quad n \to \infty \quad .$$

Proof. This is a special case of Theorem 3.2 below. It is obtained by putting $\alpha = 0$, $\beta_n = \pi - \frac{1}{n}$, and $g_n(t) = f_n(t)$ for all $t \in [0,\pi]$ and for all $n \geq 1$. q.e.d.

The next result is a generalization of Theorem 3.1, which we prove now, for future reference.

THEOREM 3.2. Let $\alpha \in [0,\pi)$, and let $g_n : [0,\pi] \to \mathbb{R}$ be such that

(i) g_n is bounded and measurable;

(ii) $0 \le g_n(t) \le g_{n+1}(t) \le f(t)$ for all $n \ge 1$, and for all $t \in [0,\pi]$;

(iii) $g_n(t) = f(t)$ for all $t \in [\alpha,\beta_n)$, where $\{\beta_n\} \subset [\alpha,\pi)$ is monotonically non-decreasing, and $\beta_n \to \pi$ as $n \to \infty$.

Then (a) there exists a unique solution (γ_n,ψ_n) of

$$\psi(s) = \frac{2\gamma}{3}\int_0^\pi G(s,t)g_n(t)\psi(t)dt \quad , \tag{3.4}$$

with $(\gamma,\psi) \in [0,\infty) \times K_0$ and $|\psi|_{C_0[0,\pi]} = 1$. Moreover, if $(\gamma,\psi) \in \mathbb{R} \times C_0[0,\pi]$ satisfies (3.4), and $\psi \neq 0$, then $|\gamma| \ge \gamma_n$.

(b) For each integer $n \ge 1$, $\gamma_{n+1} \le \gamma_n$ and $\gamma_n \to \frac{6}{\pi}$ as $n \to \infty$.

Proof. (a) Let E_n denote the support of g_n in $[0,\pi]$, and let K_n denote the reproducing cone of 'functions' in $L_2(E_n)$ which are non-negative on E_n. For $\psi \in L_2(E_n)$, define

$$L_n\psi(s) = \frac{2}{3}\int_{E_n} G(s,t)g_n(t)\psi(t)dt \quad ,$$

so that L_n is a completely continuous linear operator on $L_2(E_n)$ which is positive with respect to K_n (see Appendix). If $\psi \in K_n$ and is non-zero, then there exists an interval $(a,b) \subset [0,\pi]$ such that

$$\int_a^b g_n(t)\psi(t)dt = w > 0 \quad ,$$

where we have identified $\psi \in L_2(E_n)$ with its extension to $[0,\pi]$ as zero on $[0,\pi]\setminus E_n$. Therefore, by Theorem 2.5(c), there exists $\delta > 0$ such that

$$L_n\psi(s) = \frac{2}{3}\int_0^{\pi} G(s,t)g_n(t)\psi(t)dt$$

$$\geq \frac{2}{3}\int_a^b G(s,t)g_n(t)\psi(t)dt$$

$$\geq \left(\frac{2\delta}{3}\sin s\right)\int_a^b g_n(t)\psi(t)dt$$

$$= \frac{2\delta w}{3}\sin s \quad .$$

We have shown that L_n is u_0-bounded below, where, for $s \in E_n$, $u_0(s) = \sin s$.

To show that L_n is u_0-bounded above (for the same function $u_0 \in K_n$), it suffices to show that the operator $\mathcal{L}_n$, defined by (3.2), is v_0-bounded above on $L_2[0,\pi]$, with respect to K (the cone of non-negative 'functions' in $L_2[0,\pi]$), where $v_0 = \sin s$, for $s \in [0,\pi]$. But for $\psi \in K$ and $s \in [0,\pi]$,

$$\mathcal{L}_n\psi(s) \leq \frac{2}{3}\,|g_n|_{L_\infty[0,\pi]}\int_0^{\pi} G(s,t)\psi(t)dt \quad .$$

Therefore

$$(\mathcal{L}_n)^2\psi(s) \leq \left(\frac{2}{3}\,|g_n|_{L_\infty[0,\pi]}\right)^2 \int_0^{\pi}\int_0^{\pi} G(s,u)G(u,t)\psi(t)dudt$$

$$= \left(\frac{1}{3}\,|g_n|_{L_\infty[0,\pi]}\right)^2 2\int_0^{\pi} G_1(s,t)\psi(t)dt$$

where G_1 is given in Theorem 2.5(g). Hence

$$(\mathcal{L}_n)^2\psi(s) \leq M\sin s \quad ,$$

for all $s \in [0,\pi]$ and for some $M > 0$, where M depends on ψ.

We conclude that L_n is a completely continuous, u_0-positive, linear operator on $L_2(E_n)$, with respect to the cone K_n. By Theorem A3, there exists a unique $(\gamma_n,\bar{\psi}_n) \in (0,\infty) \times K_n$ such that $|\bar{\psi}_n|_{L_2(E_n)} = 1$, and such that $(\gamma_n,\bar{\psi}_n)$ satisfies the equation

$$\psi(s) = \frac{2\gamma}{3} \int_{E_n} G(s,t) g_n(t) \psi(t) dt \quad ,$$

for all $s \in E_n$. Moreover, γ_n is a lower bound for the absolute values of the characteristic values of L_n. If $\psi_n : [0,\pi] \to \mathbb{R}$ is defined by putting

$$\psi_n(s) = \frac{2}{3} \int_{E_n} G(s,t) g_n(t) \bar{\psi}_n(t) dt \quad ,$$

for $s \in [0,\pi]$, then it is immediate that (γ_n,ψ_n) satisfies (3.4) and that the conclusion of (a) holds.

(b) For each integer $n \geq 1$,

$$\begin{aligned} \psi_n(s) &= \frac{2\gamma_n}{3} \int_0^\pi G(s,t) g_n(t) \psi_n(t) dt \\ &\leq \frac{2\gamma_n}{3} \int_0^\pi G(s,t) g_{n+1}(t) \psi_n(t) dt \quad , \end{aligned}$$

whence $\gamma_{n+1} \leq \gamma_n$ by Theorem A1. Next we show that $\gamma_n \geq \frac{6}{\pi}$ for all integers $n \geq 1$. By Fubini's theorem and Theorem 2.5(e), it follows from (3.4) that

$$\begin{aligned} \frac{1}{2} \int_0^\pi \tan \frac{s}{2} \psi_n(s) ds &= \frac{\gamma_n}{6} \int_0^\pi t\, g_n(t) \psi_n(t) dt \\ &\leq \frac{\gamma_n}{12} \int_0^\pi t \sec \frac{t}{2} \psi_n(t) dt \\ &\leq \frac{\pi\gamma_n}{12} \int_0^\pi \tan \frac{t}{2} \psi_n(t) dt \quad . \end{aligned}$$

Hence $\gamma_n \geq \frac{6}{\pi}$, and so $\gamma_n \downarrow \gamma$ for some $\gamma \geq \frac{6}{\pi}$.

To show that $\gamma = \frac{6}{\pi}$, we use a variational characterization of γ_n. Let H denote the space of odd functions u in $L^2[-\pi,\pi]$ where

$$u \sim \sum_{k=1}^{\infty} a_k \sin ks$$

and

$$\sum_{k=1}^{\infty} k\, a_k^2 < \infty \quad .$$

For $u, v \in H$, put

$$\langle u,v \rangle = \sum_{k=1}^{\infty} k\, a_k b_k \quad , \tag{3.5}$$

if $u \sim \sum_{k=1}^{\infty} a_k \sin ks$, and $v \sim \sum_{k=1}^{\infty} b_k \sin ks$. The inner-product (3.5) makes H a Hilbert space which is compactly embedded in $L_2(-\pi,\pi)$. (H is the Hilbert space of odd functions in the fractional order Sobolev space $H^{1/2}(-\pi,\pi)$.) For $u \in H$, define

$$A_n u(s) = \frac{1}{3}\int_{-\pi}^{\pi} G(s,t) g_n(t) u(t)\,dt \quad ,$$

where g_n has been extended, as an even function, to $[-\pi,\pi]$. Clearly A_n is a compact operator from H into itself, and

$$\langle A_n u,v \rangle = \frac{1}{3\pi}\int_{-\pi}^{\pi} g_n(t) u(t) v(t)\,dt$$

for all u and v in H, whence A_n is a compact, self-adjoint linear operator on H. If the function ψ_n, defined in (a), is identified with its odd extension to $[-\pi,\pi]$, then $\psi_n \in H$, and it follows, by the Rayleigh-Ritz minimax principle [44] that

$$\gamma_n^{-1} = \sup_{u \in H\setminus\{0\}} \frac{\langle A_n u,u\rangle}{\langle u,u\rangle}$$

$$= \sup_{u \in H\setminus\{0\}} \frac{\frac{1}{3\pi}\int_{-\pi}^{\pi} g_n(t)u(t)^2 dt}{\sum_{k=1}^{\infty} k\, a_k^2} ,$$

where $u \sim \sum_{k=1}^{\infty} a_k \sin ks$.

Suppose now that $u \in H\setminus\{0\}$ is such that $|u(t)^2 f(t)| \leq M$ on $(-\pi,\pi)$, for some constant $M > 0$. Then, if $g(t) = \lim_{n\to\infty} g_n(t)$ for all $t \in (-\pi,\pi)$, it follows by the Monotone Convergence Theorem that

$$\int_{-\pi}^{\pi} g_n(t)u(t)^2 dt \to \int_{-\pi}^{\pi} g(t)u(t)^2 dt < \infty$$

as $n \to \infty$. Hence, for each such $u \in H$,

$$\gamma^{-1} = \lim_{n\to\infty} \gamma_n^{-1} \geq \lim_{n\to\infty} \frac{\frac{1}{3\pi}\int_{-\pi}^{\pi} g_n(t)u(t)^2 dt}{\langle u,u\rangle}$$

$$= \frac{\frac{1}{3\pi}\int_{-\pi}^{\pi} g(t)u(t)^2 dt}{(\sum_{k=1}^{\infty} k\, a_k^2)} , \qquad (3.6)$$

where $u \sim \sum_{k=1}^{\infty} a_k \sin ks$. It suffices to show that the value of (3.6) can be made arbitrarily close to $\pi/6$ by a judicious choice of u. We do this by explicit calculation.

Let $u_m(s) = \sum_{k=1}^{m} \frac{\sin k(\pi-s)}{k}$, $s \in [-\pi,\pi]$. Then $u_m \in H$ and $|u_m(t)^2 f(t)|$ is bounded on $(-\pi,\pi)$. Furthermore, for $s \in [0,\pi]$,

$$u_m(s) = \int_0^{(\pi-s)} (\sum_{k=1}^{m} \cos kt)dt = \int_0^{(\pi-s)} \left(\frac{\sin(m+\frac{1}{2})t}{2\sin\frac{1}{2}t} - \frac{1}{2}\right)dt .$$

Hence

$$|u_m(s) - \int_0^{(\pi-s)} \frac{\sin mt}{t} dt| \tag{3.7}$$

$$= \frac{1}{2} \left| \int_0^{(\pi-s)} \{\cos mt - 1 + \sin mt(\cot \frac{t}{2} - \frac{2}{t})\} dt \right|$$

$$\leq M(\pi-s), \quad s \in [0,\pi] \quad , \tag{3.8}$$

where M is some absolute constant, independent of m. The use of (3.8) yields that

$$u_m(s)^2 g(s) \geq \{(\int_0^{\pi-s} \frac{\sin mt}{t} dt)^2 - 2M(\pi-s) \int_0^{\pi-s} \frac{\sin mt}{t} dt$$

$$- M^2(\pi-s)^2\} g(s)$$

$$\geq (\int_0^{\pi-s} \frac{\sin mt}{t} dt)^2 g(s) - M', \quad s \in [0,\pi] \quad ,$$

where M' is an absolute constant independent of m (this follows since $g(s) \leq \frac{1}{2} \sec \frac{s}{2}$ on $(-\pi,\pi)$).

From the assumptions of the theorem, it follows that $g(s) = f(s)$, $s \in [\alpha,\pi)$, and so there exists a constant M'', independent of m, such that for all m

$$\int_{-\pi}^{\pi} u_m(s)^2 g(s) ds \geq 2 \int_\alpha^\pi (\int_0^{\pi-s} \frac{\sin mt}{t} dt)^2 f(s) ds - M''$$

$$\geq 2 \int_\alpha^\pi (\int_0^{\pi-s} \frac{\sin mt}{t} dt)^2 \frac{ds}{(\pi-s)} - M''$$

$$= 2 \int_0^{\pi-\alpha} (\int_0^s \frac{\sin mt}{t} dt)^2 \frac{1}{s} ds - M''$$

$$= 2 \int_0^{\pi-\alpha} (\int_0^{ms} \frac{\sin t}{t} dt)^2 \frac{1}{s} ds - M''$$

$$= 2 \int_0^{m(\pi-\alpha)} (\int_0^s \frac{\sin t}{t} dt)^2 \frac{1}{s} ds - M'' \quad .$$

For each $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$\frac{\pi}{2} - \varepsilon \leq \int_0^s \frac{\sin t}{t} dt \leq \frac{\pi}{2}$$

for all $s \geq K_\varepsilon$. If $m \geq K_\varepsilon/(\pi-\alpha)$, it therefore follows that

$$\int_{-\pi}^{\pi} g(s) u_m(s)^2 ds \geq 2 \int_0^{K_\varepsilon} \left(\int_0^s \frac{\sin t}{t}\right)^2 \frac{1}{s} ds + 2 \int_{K_\varepsilon}^{m(\pi-\alpha)} \left(\int_0^s \frac{\sin t}{t} dt\right)^2 \frac{1}{s} ds - M''$$

$$\geq 2 \left(\frac{\pi}{2} - \varepsilon\right)^2 \ln m - M''' \quad . \tag{3.9}$$

Here M''' is a constant, which depends on ε and α, but which is independent of m, for all m sufficiently large. Since $u_m \sim \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} \sin ks$, equations (3.6) and (3.9) yield

$$\gamma^{-1} \geq \frac{\frac{2}{3\pi}\left(\frac{\pi}{2} - \varepsilon\right)^2 \ln m - M'''}{\sum_{k=1}^{m} \frac{1}{k}} \quad .$$

From the fact that M''' is independent of m, it follows, by letting $m \to \infty$, that

$$\gamma^{-1} \geq \frac{2}{3\pi}\left(\frac{\pi}{2} - \varepsilon\right)^2 \quad .$$

Since this is true for any $\varepsilon > 0$, $\gamma^{-1} \geq \frac{\pi}{6}$. q.e.d.

3.2. The existence of approximate solitary waves

We say that $\theta^{(n)}$ corresponds to an approximate solitary wave if $(\mu^{(n)}, \theta^{(n)}) \in (0,\infty) \times K_0$ and

$$\theta^{(n)}(s) = \frac{2}{3} \int_0^{\pi} G(s,t) \frac{f_n(t) \sin \theta^{(n)}(t)}{\frac{1}{\mu^{(n)}} + \int_0^t f_n(w) \sin \theta^{(n)}(w) dw} dt \quad . \tag{3.10}$$

Because the sine function takes negative values for positive values of its argument, there are difficulties in using the right-hand side of (3.10) to define a positive operator on $(0,\infty) \times K_0$. This difficulty may be removed by considering equation (3.12) below, in place of (3.10), and by then appealing to Theorem 3.3 for justification of this procedure. So let J be defined as in (1.28), and define $A_n : (0,\infty) \times K_0 \to K_0$ as follows:

$$A_n(\mu,u) = \frac{2}{3}\int_0^{\pi} G(s,t)\,\frac{f_n(t)\sin(Ju(t))}{\frac{1}{\mu} + \int_0^t f_n(w)\sin(Ju(w))\,dw}\,dt \tag{3.11a}$$

if $(\mu,u) \in (0,\infty) \times K_0$, and

$$A_n(0,u) = 0 \tag{3.11b}$$

for all $u \in K_0$. It is apparent that

$$u = A_n(\mu,u) \tag{3.12}$$

is an operator equation in $[0,\infty) \times K_0$ to which Theorem A4 may be applied. First, though, we establish the connection between (3.12) and (3.10).

THEOREM 3.3. Let $(\mu^{(n)},\theta^{(n)}) \in (0,\infty) \times K_0$ be a non-trivial solution of (3.12). Then

(a) $\mu^{(n)} > \lambda_n$, where λ_n is defined in Theorem 3.1 (a),

(b) $\theta^{(n)} \in C^{1,\alpha}[0,\pi]$ for all α, $0 < \alpha < 1$,

(c) $0 < \theta^{(n)}(s) < \pi$ for all $s \in (0,\pi)$,

and (d) $0 < \theta^{(n)}(s) < \frac{\pi}{2}$ for all $s \in (0,\pi - \frac{1}{n}]$.

Hence every solution in $(0,\infty) \times K_0$ of (3.12) is a solution of (3.10). Moreover, every solution $(\mu^{(n)},\theta^{(n)}) \in (0,\infty) \times K_0$ of (3.10), with $|\theta^{(n)}|_{C[0,\pi]} \leq \pi$, has

$$|\theta^{(n)}|_{C[0,\pi - \frac{1}{n}]} < \frac{\pi}{2}.$$

Proof. For convenience with notation, we shall write (μ,θ) instead of $(\mu^{(n)},\theta^{(n)})$, for a non-trivial solution of (3.12), in the proof of this theorem.

(a) If (λ_n,φ_n) is as in Theorem 3.1(a), then, by Fubini's theorem and (3.12), it follows that

$$\lambda_n \int_0^\pi \theta(s) f_n(s)\varphi_n(s)ds = \int_0^\pi \frac{\varphi_n(t) f_n(t)\sin(J\theta(t))}{\frac{1}{\mu} + \int_0^t f_n(w)\sin(J\theta(w))dw} dt$$

$$< \mu \int_0^\pi \varphi_n(t) f_n(t)\theta(t)dt \quad ,$$

whence (a) follows.

(b) Now, and elsewhere when necessary, we identify θ and its extension as an odd function to $[-\pi,\pi]$. It is then immediate, by Fubini's theorem, that

$$\theta = \underset{\sim}{C}\tau \tag{3.13}$$

where τ is the continuously differentiable, even function defined by

$$\tau(t) = \frac{1}{3}\,\ell n\left(\frac{1}{\mu} + \int_0^\pi f_n(w)\sin(J\theta(w))dw\right)$$

$$- \frac{1}{3}\,\ell n\left(\frac{1}{\mu} + \int_0^t f_n(w)\sin(J\theta(w))dw\right) \tag{3.14}$$

for all $t \in [-\pi,\pi]$. Theorem 2.2(a) ensures that $\theta \in C^{0,\alpha}[-\pi,\pi]$ for all $\alpha \in (0,1)$. Since

$$3\theta' = -\underset{\sim}{C}\Psi \quad ,$$

where Ψ is the odd function given by

$$\Psi(t) = \frac{f_n(t)\sin(J\theta(t))}{\frac{1}{\mu} + \int_0^t f_n(w)\sin(J\theta(w))dw}$$

for $t \in (-\pi,\pi)$, it follows that $\Psi \in C^{0,\alpha}[-\pi,\pi]$ for all $\alpha \in (0,1)$, whence $\theta \in C^{1,\alpha}[-\pi,\pi]$ for all $\alpha \in (0,1)$.

(c) Now suppose that

$$\theta \sim \sum_{k=1}^{\infty} a_k \sin ks \quad . \tag{3.15}$$

Then, in polar coordinates,

$$\tilde{\theta}(r,t) = \sum_{k=1}^{\infty} a_k r^k \sin kt \quad , \tag{3.16}$$

$(r,t) \in [0,1] \times (-\pi,\pi]$, defines a harmonic function $\tilde{\theta}$ on the open unit ball B in $\mathbb{R}^2$. Furthermore, $\tilde{\theta}(1,t) = \theta(t)$ for $t \in (-\pi,\pi]$ and

$$\left.\frac{\partial\tilde{\theta}}{\partial r}\right|_{(r,t)} = \sum_{k=1}^{\infty} k\, a_k r^{k-1} \sin kt \quad ,$$

for $(r,t) \in [0,1) \times (-\pi,\pi]$. By (b) and Theorems 2.1 and 2.4, it follows that

$$\left.\frac{\partial\tilde{\theta}}{\partial r}\right|_{(1,t)} = \lim_{r\to 1} \left.\frac{\partial\tilde{\theta}}{\partial r}\right|_{(r,t)} = \sum_{k=1}^{\infty} k\, a_k \sin kt$$

$$= \frac{1}{3}\, \frac{f_n(t)\sin(J\theta(t))}{\frac{1}{\mu} + \int_0^t f_n(w)\sin(J\theta(w))dw} \quad .$$

The function $\tilde{\theta}$ attains its maximum on ∂B at some point $(1,t_0)$, $t_0 \in (0,\pi)$, and at such a point $\frac{\partial\tilde{\theta}}{\partial r} > 0$. Thus $\sin(J\theta(t_0)) \neq 0$, and so (c) is established.

(d) From (c) it follows that $J\theta = \theta$, whence, by (3.13) and (3.14),

$$\theta = \underset{\sim}{C}\tau \quad , \tag{3.17a}$$

and

$$\underset{\sim}{C}\theta = -\tau + a_0 \quad , \tag{3.17b}$$

where now

$$\tau(t) = \frac{1}{3}\,\ell n\Big(\frac{1}{\mu} + \int_0^{\pi} f_n(w)\sin\theta(w)dw\Big) -$$

$$\frac{1}{3}\,\ell n\Big(\frac{1}{\mu} + \int_0^{t} f_n(w)\sin\theta(w)dw\Big) \quad , \tag{3.18}$$

and

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tau(t)dt \quad . \tag{3.19}$$

The point in B whose polar coordinates are (r,t) may be identified with the point re^{it} in the open unit ball in the complex plane. Thus we can define an analytic function on B by putting

$$\tilde{\omega}(re^{it}) = a_0 + \sum_{k=1}^{\infty} a_k r^k e^{ikt} \quad , \tag{3.20}$$

where the sequence $\{a_k\}$ is determined by (3.15) and (3.19). Then $\tilde{\omega} = \tilde{\tau} + i\tilde{\theta}$, where $\tilde{\theta}$ is given by (3.16),

$$\tilde{\tau}(re^{it}) = a_0 + \sum_{k=1}^{\infty} a_k r^k \cos kt \quad , \tag{3.21}$$

and

$$\tilde{\tau}(e^{it}) = \tau(t) \quad , \quad \tilde{\theta}(e^{it}) = \theta(t) \tag{3.22}$$

for all $t \in (-\pi,\pi]$.

Recall, from (b), that $\theta \in C^{1,\alpha}[-\pi,\pi]$ for all $\alpha \in (0,1)$, and so Theorem 2.1 ensures that $\tilde{\theta} \in C^{1,\alpha}(\bar{B})$. Equation (3.18) implies that $\tau \in C^{1,\alpha}[-\pi,\pi]$, whence $\tilde{\tau} \in C^{1,\alpha}(\bar{B})$. Hence

$$\tilde{\omega} = \tilde{\tau} + i\tilde{\theta} \in C^{1,\alpha}(\bar{B}) \quad . \tag{3.23}$$

Since $\tilde{\theta}(re^{i0}) = \tilde{\theta}(re^{i\pi}) = 0$ for $r \in [0,1]$, and since $\theta(t) > 0$ on $(0,\pi)$, it follows by the maximum principle that

$$\left.\frac{\partial\tilde{\theta}}{\partial t}\right|_{re^{i\pi}} < 0 \quad , \quad r \in (0,1) \quad . \tag{3.24}$$

From (3.16) and (3.21) it is evident that

$$r\,\frac{\partial\tilde{\tau}}{\partial r} = \frac{\partial\tilde{\theta}}{\partial t} \quad , \quad r \in (0,1] \tag{3.25}$$

and so, for each $r \in (0,1]$, equation (3.24) gives

$$\left.\frac{\partial\tilde{\tau}}{\partial r}\right|_{re^{i\pi}} < 0 \tag{3.26}$$

Let $B' = \{re^{it} \in B : t \neq \pi\}$, and define an analytic function Z on B' by

$$Z(\xi) = \frac{1}{3(\frac{1}{\mu} + \int_0^\pi f_n(w)\sin\theta(w)dw)} \int_0^\xi \left(\frac{i\zeta^{-1/2}}{1+\zeta}\right)\exp(\tilde{\omega}(\zeta))d\zeta \quad .$$

The imaginary part of Z defines a harmonic function on B' which we denote by Y:

$$Y(r,t) = \operatorname{Im} Z(re^{it}) \quad .$$

Since $\tilde{\omega}$ is real on the negative real axis, Y can be extended continuously to B by putting

$$Y(r,\pi) = 0, \quad 0 \leq r < 1 \quad . \tag{3.27}$$

Now define a function $P : B' \to \mathbb{R}$ by

$$P(re^{it}) = -\frac{1}{2}\exp(-2\tilde{\tau}(re^{it})) - Y(r,t) \quad .^*$$

Since both $\tilde{\tau}$ and Y are harmonic in B', it follows that P is super-harmonic

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*It is of interest to note that in this approximate problem, P is analogous to the pressure, $-\frac{1}{2}q^2 - gy$, at a point (x,y) of a real flow domain in the z-plane. The underlying motivation for the argument which follows is that, for an actual solitary wave flow, the pressure is a super-harmonic function in the z-plane, and is a constant along the free surface. The function Z is the actual mapping from $\mathcal{D}'$ into the z-plane in this approximate situation. (Compare for example with (1.11), (1.12).)

(div grad $P \leq 0$) in B'. Clearly $P(re^{it}) = P(re^{-it})$, and so P can be extended as a continuous function on B. It follows from (3.23) that P can be extended continuously to $\bar{B}\backslash\{e^{it}\}$. Moreover, if $t \in (-\pi,\pi)$ it follows from (3.18) and (3.23) that

$$\left.\frac{\partial P}{\partial t}\right|_{e^{it}} = \tau'(t)\exp(-2\tau(t)) + \frac{\exp(\tau(t))\sin\theta(t)}{6\cos\frac{t}{2}(\frac{1}{\mu} + \int_0^\pi f_n(w)\sin\theta(w)dw)}$$

$$= \exp(\tau(t))\{\tau'(t)\exp(-3\tau(t)) + \frac{f(t)\sin\theta(t)}{3(\frac{1}{\mu} + \int_0^\pi f_n(w)\sin\theta(w)dw)}\}$$

$$= \frac{\exp(\tau(t))\sin\theta(t)\{f(t) - f_n(t)\}}{3\{\frac{1}{\mu} + \int_0^\pi f_n(w)\sin\theta(w)dw\}} .$$

So

$$\frac{\partial P}{\partial t} = 0 \quad \text{if} \quad |t| \leq \pi - \frac{1}{n} , \tag{3.28}$$

and

$$\frac{\partial P}{\partial t} \geq 0 \quad \text{if} \quad \pi - \frac{1}{n} \leq t < \pi . \tag{3.29}$$

The maximum principle for super-harmonic functions ensures that P attains its minimum on $\partial B'$. By (3.26) - (3.29) this minimum is attained at every point on the boundary portion $\{e^{it} : |t| \leq \pi - \frac{1}{n}\}$. By the strong maximum principle, therefore,

$$\left.\frac{\partial P}{\partial r}\right|_{e^{it}} < 0 \quad \text{if} \quad |t| \leq \pi - \frac{1}{n} . \tag{3.30}$$

The use of this with (3.25) yields

$$0 > \exp(-2\tau(t))\left.\frac{\partial\tilde{\tau}}{\partial r}\right|_{e^{it}} - \frac{\exp(\tau(t))\cos\theta(t)}{6\cos\frac{t}{2}(\frac{1}{\mu} + \int_0^\pi f_n(w)\sin\theta(w)dw)}$$

$$= \exp(\tau(t))\{\exp(-3\tau(t))\theta'(t) - \frac{\cos\theta(t)}{6\cos\frac{t}{2}(\frac{1}{\mu} + \int_0^\pi f_n(w)\sin\theta(w)dw)}\} ,$$

if $|t| \leq \pi - \frac{1}{n}$. Therefore if $|t| \leq \pi - \frac{1}{n}$,

$$\exp(-3\tau(t))\theta'(t) < \frac{\cos\theta(t)}{6\cos\frac{t}{2}(\frac{1}{\mu} + \int_0^\pi f_n(w)\sin\theta(w)dw)},$$

and, as a consequence, $\theta'(t) < 0$ if $\frac{\pi}{2} \leq \theta(t) < \pi$ and $|t| \leq \pi - \frac{1}{n}$. This implies that there are no t with $|t| \leq \pi - \frac{1}{n}$ for which $\theta(t) \geq \pi/2$, and (d) has been established. q.e.d.

As a result of Theorem 3.3(c), the question of the existence of non-trivial approximate solitary waves may be solved by a study of equation (3.12).

LEMMA 3.4. <u>The mapping</u> $A_n : [0,\infty) \times K_0 \to K_0$, <u>which is defined in</u> (3.11) <u>has the following properties</u>.

(a) <u>It is a continuous mapping from</u> $[0,\infty) \times K_0$ <u>into</u> K_0, <u>and maps bounded subsets of</u> $[0,\infty) \times K_0$ <u>into relatively compact subsets of</u> $C_0[0,\pi]$.

(b) <u>If</u> $L_n : C_0[0,\pi] \to C_0[0,\pi]$ <u>is the completely continuous, linear operator defined in</u> (3.2), <u>then</u>

$$(|u|_{C_0[0,\pi]})^{-1}|A_n(\mu,u) - \mu L_n u|_{C_0[0,\pi]} \to 0 \tag{3.31}$$

<u>as</u> $|u|_{C_0[0,\pi]} \to 0$, <u>uniformly for</u> μ <u>in bounded intervals of</u> $[0,\infty)$.

(c) $A_n(0,u) = A_n(\mu,0) = 0$ <u>for all</u> $(\mu,u) \in [0,\infty) \times K_0$.

<u>Proof</u>. (a) This follows immediately from Theorem 2.5(a) and (b).

(b) Equation (3.31) holds, since

$$|f_n(t)[\mu(\sin(J\theta(t)) - \theta(t)) - \mu^2\theta(t)\int_0^t f_n(w)\sin(J\theta(w))dw]|$$

is bounded by a constant multiple of $(|\theta|_{C_0[0,\pi]})^2$, uniformly for $t \in [0,\pi]$ and for μ in bounded intervals of $[0,\infty)$.

(c) This is immediate from the definition of A_n. q.e.d.

The main result of this section, on the global existence of non-trivial approximate solitary waves, now follows. Let $S_n = \{(\mu^{(n)},\theta^{(n)}) \in [0,\infty) \times K_0 : \theta^{(n)} \neq 0$ and $\theta^{(n)} = A_n(\mu^{(n)},\theta^{(n)})\} \cup \{(\lambda_n,0)\}$, where λ_n is defined as in Theorem 3.1(a).

THEOREM 3.5. <u>If the maximal connected subset of</u> S_n <u>which contains</u> $(\lambda_n,0)$ <u>is denoted by</u> C_n, <u>then</u>

(a) C_n <u>is closed and unbounded in</u> $\mathbb{R} \times C_0[0,\pi]$, <u>and</u>

(b) $\{\mu^{(n)} : (\mu^{(n)},\theta^{(n)}) \in C_n$ <u>for some</u> $\theta^{(n)} \in K_0\} = [\lambda_n,\infty)$.

<u>If</u> $(\mu^{(n)},\theta^{(n)}) \in C_n \setminus \{(\lambda_n,0)\}$, <u>then</u>

(c) $0 < \theta^{(n)}(s) < \pi$ <u>on</u> $(0,\pi)$ <u>and</u> $\theta^{(n)}(s) < \frac{\pi}{2}$ <u>on</u> $(0,\pi - \frac{1}{n}]$,

(d) $\theta^{(n)} \in C^{1,\alpha}[0,\pi]$ <u>for each</u> $\alpha \in (0,1)$, <u>and</u>

(e) <u>if</u> $\tau^{(n)}$ <u>is defined in terms of</u> $(\mu^{(n)},\theta^{(n)})$ <u>by</u> (3.18), <u>then</u> $\tau^{(n)}$ <u>and</u> $\theta^{(n)}$ <u>satisfy</u> (3.17a) <u>and</u> (3.17b).

<u>Proof</u>. (a) is a consequence of Theorem 3.1, Lemma 3.4, and Theorem A4.

(b) This follows as a consequence of (a), and the a priori bound given in Theorem 3.3.

(c),(d),(e) These results were established in Theorem 3.3. q.e.d.

COROLLARY 3.6. <u>Let</u> U <u>be a bounded, open set in</u> $\mathbb{R} \times C_0[0,\pi]$, <u>such that</u> $(\frac{6}{\pi},0) \in U$. <u>Then, by Theorem</u> 3.1(b), $(\lambda_n,0) \in U$ <u>for all</u> n <u>sufficiently large</u>. <u>For all such</u> n,

$$C_n \cap \partial U \neq \phi \quad ,$$

<u>and if</u> $(\mu^{(n)},\theta^{(n)}) \in C_n \cap \partial U$, <u>then</u> (c), (d) <u>and</u> (e) <u>of Theorem</u> 3.5 <u>hold</u>.

<u>Proof</u>. This result is a direct application of the remark after Theorem A6, in the light of Theorems 3.1 and 3.5. q.e.d.

3.3. The global existence of solitary waves

In this section, we prove the existence of an unbounded, connected set in $(\frac{6}{\pi},\infty) \times K_0$ of non-trivial solutions of (1.26), and each of these solutions corresponds to a non-trivial solitary wave (by section 1.2). Before we can do this, however, it is necessary to establish some general properties of solutions of (1.26) which lie in S, where $S = \{(\mu,\theta) \in (0,\infty) \times K_0 : (\mu,\theta)$ is a solution of (1.26) with $0 < \theta(s) < \pi$ for $s \in (0,\pi)$, and $\int_0^\pi f(t)\sin\theta(t)dt < \infty\} \cup \{(\frac{6}{\pi},0)\}$. From Theorem 1.1, it may be recalled that the set $S\backslash\{(\frac{6}{\pi},0)\}$ consists of those solutions of (1.26) which correspond to non-trivial solitary waves. If $(\mu,\theta) \in S$, then the corresponding solitary wave has Froude number F, whose value is given in terms of μ and θ by

$$\frac{\pi}{6}F^2 = \frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)dw \quad .$$

<u>Remark</u>. Though no subsequent use is made of the fact, we show in Theorem 4.1 that every non-trivial positive solution of (1.26) in $(0,\infty) \times L_2(0,\pi)$ actually lies in S.

THEOREM 3.7. <u>If</u> $(\mu,\theta) \in S\backslash\{(\frac{6}{\pi},0)\}$, <u>then</u>

(a) $\theta \in C^{1,\alpha}[a,b]$ <u>for all closed intervals</u> $[a,b] \subset [0,\pi)$ <u>and for all</u> $\alpha \in (0,1)$,

(b) $0 < \theta(s) < \frac{\pi}{2}$ <u>for all</u> $s \in (0,\pi)$,

(c) $\mu > \frac{6}{\pi}$, <u>and</u>

$$\frac{\pi}{6} \leq \frac{\pi}{6}F^2 = \frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)dw \leq M^{\dagger},$$

<u>where</u> M <u>is an absolute constant, independent of</u> $(\mu,\theta) \in S$.

†Indeed, the use of Theorems 4.6 and 4.12 in Theorem 4.8(c) ensure that $F^2 < 4$.

Proof. (a) and (b) are proved by a method similar to that used in the proof of Theorem 3.3(b) and (d). The necessary modifications are obvious, and there is no need to repeat the details here.

(c) If $(\mu,\theta) \in S\backslash\{(\frac{6}{\pi},0)\}$, then Theorem 2.5(c) ensures that $\theta(s) > \beta \sin s$ for some $\beta > 0$. So, Theorem 2.5(e), Fubini's theorem and the fact that $\frac{\pi}{2}\tan\frac{t}{2} > t\, f(t)$ for all $t \in (0,\pi)$, together yield

$$0 < \int_0^\pi \frac{1}{2}\tan\frac{s}{2}\,\theta(s)ds < \frac{\mu}{6}\int_0^\pi t\, f(t)\,\sin\theta(t)dt$$

$$< \frac{\mu\pi}{6}\int_0^\pi \frac{1}{2}\tan\frac{t}{2}\,\theta(t)dt \quad ,$$

whence $\mu > \frac{6}{\pi}$.

To prove that $F^2 \geq 1$, we proceed as follows. Let $\alpha \in (0,\pi)$ and let $\{\beta_n\} \subset (\alpha,\pi)$ be such that $\beta_n \uparrow \pi$ as $n \to \infty$. Put

$$g_n(t) = \begin{cases} 0 & \text{if } t \in [0,\alpha) , \\ f(t) & \text{if } t \in [\alpha,\beta_n] , \\ f(\beta_n) & \text{if } t \in [\beta_n,\pi) . \end{cases}$$

Then $\{g_n\}$ satisfies all the conditions of Theorem 3.2, and so there exists a sequence $\{(\gamma_n,\psi_n)\} \in (0,\infty) \times K_0$ satisfying the equation

$$\psi_n(s) = \frac{2\gamma_n}{3}\int_0^\pi G(s,t)g_n(t)\psi_n(t)dt$$

for each n, and $\gamma_n \downarrow \frac{6}{\pi}$. Multiplying (1.26) by $\gamma_n\psi_n g_n$ and integrating over $[0,\pi]$ gives

$$\gamma_n \int_0^\pi \theta(s)\psi_n(s)g_n(s)ds = \int_0^\pi \frac{\psi_n(t)f(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f(w)\sin\theta(w)dw} dt$$

$$\geq A_\alpha \int_0^\pi \frac{\psi_n(t)g_n(t)\theta(t)}{\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)dw} dt \quad ,$$

where

$$A_\alpha = \inf_{t\in[\alpha,\pi]} \frac{\sin\theta(t)}{\theta(t)} \quad .$$

Hence

$$\gamma_n\left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)dw\right) \geq A_\alpha$$

for all $n \geq 1$ and for all $\alpha \in (0,\pi)$. Since $\gamma_n \downarrow \frac{6}{\pi}$, it follows that

$$\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)dw \geq \frac{\pi}{6} A_\alpha \quad .$$

But $\theta \in K_0$, and so $A_\alpha \to 1$ as $\alpha \to \pi$. The result then follows.

To complete the proof, we must show that F^2 is bounded above. Assume, on the contrary, that S contains a sequence $\{(\mu_n,\theta_n)\}$ such that $\frac{1}{\mu_n} + \int_0^\pi f(w)\sin\theta_n(w)dw \to \infty$ as $n \to \infty$. Let

$$r(t) = f(t) - \frac{1}{2}\tan\frac{t}{2}, \quad t \in [0,\pi) \quad . \tag{3.32}$$

Then $r(s) \leq \frac{1}{2}\cos\frac{s}{2}$, and by Theorem 2.5(e),

$$\int_0^\pi f(s)\theta_n(s)ds = \int_0^\pi \theta_n(s)r(s)ds + \frac{1}{6}\int_0^\pi \frac{t\,f(t)\sin\theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f(w)\sin\theta_n(w)dw}\,dt$$

$$= \int_0^\pi \theta_n(s)r(s)ds + \frac{\pi}{6}\,\ell n\left(\frac{1}{\mu_n} + \int_0^\pi f(w)\sin\theta_n(w)dw\right)$$

$$- \frac{1}{6}\int_0^\pi \ell n\left(\frac{1}{\mu_n} + \int_0^t f(w)\sin\theta_n(w)dw\right)dt \quad . \tag{3.33}$$

Since $\int_0^t f(w)\sin\theta_n(w)dw \leq \frac{\pi}{2}\,\ell n\left(\frac{\pi}{\pi - t}\right)$, it follows that, if $\{\mu_n\}$ is bounded, then there exists a constant M, independent of n, such that

$$\int_0^\pi f(s)\theta_n(s)ds \leq M + \frac{\pi}{6}\,\ell n\left(\frac{1}{\mu_n} + \int_0^\pi f(w)\sin\theta_n(w)dw\right) \quad ,$$

and this is a contradiction. Hence the sequence $\{\mu_n\}$ is unbounded.

We may therefore suppose, without loss of generality, that $\mu_n \to \infty$ as $n \to \infty$. Then

$$\theta_n(s) = \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f(t)\sin\theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f(w)\sin\theta_n(w)dw}\,dt$$

$$\geq \frac{2}{3}\int_0^\pi G(s,t)\,\frac{\sin\theta_n(t)}{\frac{2}{\mu_n} + \int_0^t \sin\theta_n(w)dw}\,dt \quad .^{*} \tag{3.34}$$

Hence, by (b),

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*This follows from the observation that if $\alpha \geq 0$ and $u \geq 0$, then the fact that $f \geq 1/2$ and is increasing gives

$$\frac{f(t)u(t)}{\alpha + \int_0^t f(w)u(w)dw} \geq \frac{u(t)}{2\alpha + \int_0^t u(w)dw} \quad .$$

$$\int_0^\pi \theta_n(s)\sin s\, ds \geq \frac{1}{3}\int_0^\pi \frac{\sin t \sin \theta_n(t)}{\frac{2}{\mu_n} + \int_0^\pi \sin \theta_n(w)dw}\, dt$$

$$\geq \frac{2}{3\pi} \frac{\int_0^\pi \sin t\, \theta_n(t)dt}{\frac{2}{\mu_n} + \int_0^\pi \sin \theta_n(w)dw} ,$$

and so

$$\frac{2}{\mu_n} + \int_0^\pi \sin \theta_n(w)dw \geq \frac{2}{3\pi} . \tag{3.35}$$

Since $\mu_n \to \infty$, it follows from (3.35) that there exists $\alpha > 0$ and $N > 0$ such that

$$\int_0^\pi \theta_n(w)dw \geq \alpha > 0 \quad \text{for all} \quad n \geq N .$$

Since, by part (a), $\{\theta_n\}$ is bounded in $L_2(0,\pi)$, it may be supposed, without loss of generality, to be weakly convergent in $L_2(0,\pi)$ to θ say, where $\theta \neq 0$. Hence there exists a closed interval $[a,b] \subset (0,\pi)$ and real numbers α',N' such that

$$\int_a^b \theta_n(t)dt \geq \alpha' > 0 \quad \text{for all} \quad n \geq N' . \tag{3.36}$$

By (3.34) and (b),

$$\theta_n(s) \geq \frac{4}{3\pi(\frac{2}{\mu_n} + \pi)} \int_0^\pi G(s,t)\theta_n(t)dt$$

$$\geq \frac{4}{3\pi(\frac{2}{\mu_n} + \pi)} \int_a^b G(s,t)\theta_n(t)dt$$

$$\geq \frac{4\ \delta\alpha'}{3\pi(\frac{2}{\mu_n} + \pi)} \sin s , \tag{3.37}$$

for all $s \in (0,\pi)$, where δ is chosen to correspond to the closed interval $[a,b]$ given in (3.36), as in Theorem 2.5(c). We have shown that $\theta_n(s)$ is bounded below by a constant multiple of $\sin s$, where the constant is independent of n. Now if follows from (3.33) that

$$\int_0^\pi f(s)\theta_n(s)ds \le M + \frac{\pi}{6}\ell n(\frac{1}{\mu_n} + \int_0^\pi f(w)\sin\theta_n(w)dw) \quad ,$$

which is a contradiction as before. The proof is now complete. q.e.d.

The next result is crucial in the proof of the global existence of finite-amplitude solitary water waves.

THEOREM 3.8. <u>The set</u> S <u>is closed, and bounded subsets of</u> S <u>are relatively compact in the topology of</u> $\mathbb{R} \times C_0[0,\pi]$.

<u>Proof</u>. If $\{(\mu_n,\theta_n)\} \subset S$ is a bounded sequence in $\mathbb{R} \times C_0[0,\pi]$, then it has a subsequence $\{(\mu_{n(k)},\theta_{n(k)})\}$ with the property that

$$\mu_{n(k)} \to \mu \quad , \tag{3.38a}$$

$$\theta_{n(k)} \rightharpoonup \theta \quad , \tag{3.38b}$$

$$\sin\theta_{n(k)} \rightharpoonup \sigma \quad . \tag{3.38c}$$

Here $\mu \in \mathbb{R}$, θ and σ are in $L_2(0,\pi)$ and $\rightharpoonup$ denotes weak L_2-convergence. For the proof of the theorem, it will suffice to show that $(\mu,\theta) \in S$, and that

$$(\mu_{n(k)},\theta_{n(k)}) \to (\mu,\theta) \quad \text{in } \mathbb{R} \times C_0[0,\pi] \tag{3.39}$$

as $k \to \infty$. To simplify the notation, we will denote the subsequence satisfying (3.38) by $\{(\mu_n,\theta_n)\}$, and, when necessary, we will identify θ_n and its extension as an odd function to the interval $[-\pi,\pi]$. The proof now falls into a number of convenient stages.

(A) We first show that $(\mu_n, \theta_n) \to (\mu, \theta)$ in $\mathbb{R} \times L_2(0,\pi)$, and that $\sigma(t) = \sin\theta(t)$ almost everywhere on $(0,\pi)$.

For integers n and $k \geq 1$, the use of Fubini's theorem and (1.26) yield

$$\int_{-\pi}^{\pi} \theta_n(s)\sin ks\, ds = -\frac{1}{3}\int_{-\pi}^{\pi} \cos kt(\ln(\frac{1}{\mu_n} + \int_0^t f(w)\sin\theta_n(w)dw))dt \quad .$$

Hence, if we set

$$\rho_n(t) = -\frac{1}{3}\ln(\frac{1}{\mu_n} + \int_0^t f(w)\sin\theta_n(w)dw) \tag{3.40}$$

for all $t \in [-\pi,\pi]$, and

$$a_0^{(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho_n(t)dt \quad ,$$

then we have shown that

$$\theta_n = \underset{\sim}{C}\rho_n \tag{3.41}$$

and

$$\underset{\sim}{C}\theta_n = a_0^{(n)} - \rho_n \quad . \tag{3.42}$$

Since $\sin\theta_n \rightharpoonup \sigma$ there follows that $\rho_n(t) \to \rho(t)$ for all $t \in (-\pi,\pi)$, where

$$\rho(t) = -\frac{1}{3}\ln(\frac{1}{\mu} + \int_0^t f(w)\sigma(w)dw) \quad . \tag{3.43}$$

By Theorem 3.6, $\{\mu_n\}$ is bounded in $[\frac{6}{\pi},\infty)$, the corresponding sequence of Froude numbers is bounded in $[1,\infty)$, and $\int_0^t f(w)\sin\theta_n(w)dw \leq \frac{\pi}{2}\ln(\frac{\pi}{\pi-t})$. From the Dominated Convergence Theorem, it follows that $\rho_n \to \rho$ in $L_2(-\pi,\pi)$ as $n \to \infty$. From (3.41) and Theorem 2.3, it follows immediately that $\theta_n \to \theta$ in $L_2(-\pi,\pi)$ as $n \to \infty$. Hence there exists a subsequence $\{\theta_{n(k)}\}$ of $\{\theta_n\}$ such that $\theta_{n(k)} \to \theta$ almost everywhere on $(-\pi,\pi)$, whence $\sigma(t) = \sin\theta(t)$ for almost all $t \in (0,\pi)$.

(B) We now prove that $\theta_n \to \theta$ in $C[0,\pi-\delta]$ for each $\delta \in (0,\pi)$, and consequently that $\theta(s) \leq \pi/2$ on $[0,\pi)$.

Since $\{\rho_n\}$ is bounded in $C^{0,\alpha}[0,\pi-\delta]$ for each $\delta \in (0,\pi)$ and $\alpha \in (0,1)$, Theorem 2.4(b) and (3.41) ensure that the same is true for $\{\theta_n\}$. Hence $\{\theta_n\}$ is relatively compact in $C[0,\pi-\delta]$, and since $\theta_n \to \theta$ in $L^2(0,\pi)$, it follows that $\theta_n \to \theta$ in $C[0,\pi-\delta]$, for each $\delta \in (0,\pi)$. Therefore $\theta(s) \leq \frac{\pi}{2}$ since the same holds for each θ_n, by Theorem 3.7(b).

(C) It follows by Fatou's lemma and Theorem 3.7(c) that

$$\frac{1}{\mu} + \int_0^\pi f(t)\sin\theta(t)dt < \infty.$$

(D) The next step is to prove that (μ,θ) is a solution of (1.26), and that $\theta \in K_0$.

For each <u>fixed</u> $s \in [0,\pi)$

$$\theta_n(s) = \frac{2}{3}\int_0^\pi G(s,t) \frac{f(t)\sin\theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f(w)\sin\theta_n(w)dw} dt \quad . \tag{3.44}$$

Theorem 2.5(f) ensures that for each fixed $s \in [0,\pi)$, $G(s,t)f(t)$ is an integrable function of t. It then follows from the Dominated Convergence theorem that the right-hand side of (3.44) converges to

$$\frac{2}{3}\int_0^\pi G(s,t) \frac{f(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f(w)\sin\theta(w)dw} dt \quad ,$$

and so, from (B), that θ satisfies (1.26) on $[0,\pi)$. Since $\theta \geq 0$, it remains to show that $\lim_{s\to\pi} \theta(s) = 0$. But we know from Theorem 2.5(f) that, for $s \in [0,\pi)$,

$$\theta(s) = \frac{2}{3}\int_0^{\pi} G(s,t)\,\frac{\dfrac{f(t)\sin\theta(t)}{t}}{\dfrac{1}{\mu} + \displaystyle\int_0^t f(w)\sin\theta(w)\,dw}\,dt$$

$$\leq \frac{2\mu}{3}\int_0^{\pi} G(s,t)f(t)\theta(t)dt$$

$$= \frac{\mu}{3\pi}\int_0^{\pi}\left(\ln\left|\frac{2\pi-s-t}{s-t}\right|\right)f(t)\theta(t)dt + \frac{2\mu}{3}\int_0^{\pi} h(s,t)f(t)\theta(t)dt \quad ,$$

where

$$\frac{2\mu}{3}\int_0^{\pi} h(s,t)f(t)\theta(t)dt \leq \frac{\mu\pi}{3}\int_0^{\pi}\frac{h(s,t)}{(\pi-t)}\,\theta(t)dt$$

$$\leq \left(\frac{\mu\pi A}{3}\int_0^{\pi}\theta(t)dt\right)(\pi-s) \quad ,$$

A being given in Theorem 2.5(f). Therefore,

$$\theta(s) \leq \frac{\mu}{3\pi}\,\chi(s) + O(\pi-s) \quad , \quad s \in [0,\pi) \quad , \tag{3.45}$$

where

$$\chi(s) = \int_0^{\pi}\left(\ln\left|\frac{2\pi-s-t}{s-t}\right|\right)f(t)\theta(t)dt \quad ,$$

for all $s \in [0,\pi)$. Moreover, for $s \in [0,\pi)$,

$$\chi(s) \leq \frac{\pi}{2}\int_0^{\pi}\left(\ln\left|\frac{2\pi-s-t}{s-t}\right|\right)\frac{\theta(t)}{(\pi-t)}\,dt$$

$$= \frac{\pi}{2}\int_0^{\pi/(\pi-s)}\left(\ln\left|\frac{1+t}{1-t}\right|\right)\frac{\theta(\pi-(\pi-s)t)}{t}\,dt$$

$$\leq \frac{\pi}{2}\left\{\int_0^{L}\left(\ln\left|\frac{1+t}{1-t}\right|\right)\frac{\theta(\pi-(\pi-s)t)}{t}\,dt + \right.$$

$$\left.\int_L^{\pi/(\pi-s)}\left(\ln\left|\frac{1+t}{1-t}\right|\right)\frac{\theta(\pi-(\pi-s)t)}{t}\,dt\right\}$$

$$\leq \frac{\pi}{2}\{\int_0^L (\ell n|\frac{1+t}{1-t}|)^2 \frac{\pi}{2t}\, dt\}^{1/2}\{\int_0^L \frac{\theta(\pi-(\pi-s)t)}{t}\, dt\}^{1/2}$$

$$+ \frac{\pi^2}{4}\int_L^\infty (\ell n|\frac{1+t}{1-t}|)\frac{1}{t}\, dt$$

$$\leq M\{\int_0^{L(\pi-s)} \frac{\theta(\pi-t)}{t}\, dt\}^{1/2} + \frac{\pi^2}{4}\int_L^\infty (\ell n|\frac{1+t}{1-t}|)\frac{1}{t}\, dt \quad ,$$

where M is a constant independent of s, and $L \in (0,\pi/(\pi-s))$.

Let $\varepsilon > 0$, and choose L_ε such that

$$\frac{\pi^2}{4}\int_{L_\varepsilon}^\infty \frac{1}{t}(\ell n|\frac{1+t}{1-t}|)\,dt < \frac{\varepsilon}{2} \quad .$$

Then, for all $s \in ((L_\varepsilon-1)\pi/L_\varepsilon,\pi)$, there results that

$$\chi(s) \leq M\{\int_0^{L_\varepsilon(\pi-s)} \frac{\theta(\pi-t)}{t}\, dt\}^{1/2} + \frac{\varepsilon}{2}$$

$$= M\{\int_{\pi-L_\varepsilon(\pi-s)}^{\pi} \frac{\theta(t)}{(\pi-t)}\, dt\}^{1/2} + \frac{\varepsilon}{2} \quad , \tag{3.46}$$

and, by (C), it follows that

$$\limsup_{s\to 0} \chi(s) \leq \frac{\varepsilon}{2} \quad ,$$

whence $\lim_{s\to 0} \chi(s) = 0$. Finally it follows from (3.45) that $\theta \in K_0$.

(E) We now prove that $(\mu,\theta) \in S$. In the light of (C) and (D) above, it suffices to show that if $\theta = 0$, then $\mu = \frac{6}{\pi}$.

Suppose that $\theta = 0$. Then $(\mu_n,\theta_n) \to (\mu,0)$ in $\mathbb{R} \times L_2(0,\pi)$ and also in $\mathbb{R} \times C[0,\pi-\delta]$, for each $\delta \in (0,\pi)$. For each integer $m \geq 1$, put

$$g_m(t) = \begin{cases} f(t) & , 0 \le t \le \pi - \frac{1}{m} , \\ 0 & , \pi - \frac{1}{m} < t \le \pi . \end{cases}$$

Then the sequence $\{g_m\}$ satisfies the conditions of Theorem 3.2, and so there exists a sequence $\{(\gamma_m, \psi_m)\} \subset (0,\infty) \times (K_0 \backslash \{0\})$ such that

$$\psi_m = \frac{2\gamma_m}{3} \int_0^\pi G(s,t) g_m(t) \psi_m(t) dt \tag{3.47}$$

and $\gamma_m \downarrow \frac{6}{\pi}$. Without loss of generality, we may assume that $\theta_n \neq 0$ for each positive integer n, and so that $\theta_n(s) > 0$ on $(0,\pi)$ for each n. (If $\theta_n = 0$ for an infinite number of integers n, then, from the definition of S, $\mu_n = \frac{6}{\pi}$ for an infinite number of n, and the result is immediate.)

If m and n are positive integers, then, by (1.26), (3.47) and Fubini's theorem,

$$0 < \gamma_m \int_0^\pi \psi_m(s) g_m(s) \theta_n(s) ds$$

$$= \int_0^\pi \frac{\psi_m(t) f(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f(w) \sin \theta_n(w)} dt$$

$$\ge \int_0^\pi \frac{\psi_m(t) g_m(t) \sin \theta_n(t)}{\frac{1}{\mu_n} + \int_0^t g_m(w) \sin \theta_n(w) dw} dt \quad *$$

---

*
$$\frac{g_m(t) u(t)}{\alpha + \int_0^t g_m(w) u(w) dw} \le \frac{f(t) u(t)}{\alpha + \int_0^t f(w) u(w) dw}$$

for all $\alpha > 0$ and $u \ge 0$.

$$\geq B_{m,n}\left\{\frac{\int_0^{\pi}\psi_m(t)g_m(t)\theta_n(t)\,dt}{\frac{1}{\mu_n}+\int_0^{\pi}g_m(w)\sin\theta_n(w)\,dw}\right\},$$

where

$$B_{m,n} = \inf_{t\in[0,\pi-\frac{1}{m}]}\frac{\sin\theta_n(t)}{\theta_n(t)} .$$

Hence, for positive integers m and n,

$$\gamma_m\left(\frac{1}{\mu_n}+\int_0^{\pi}g_m(w)\sin\theta_n(w)\,dw\right) \geq B_{m,n} . \tag{3.48}$$

For each fixed integer m, $B_{m,n} \to 1$ as $n \to \infty$, since $\theta_n \to 0$ in $C[0,\pi-\frac{1}{m}]$, whence (3.48) gives

$$\frac{\gamma_m}{\mu} \geq 1$$

for each positive integer m. But $\gamma_m \downarrow \frac{6}{\pi}$, and so

$$\mu \leq \frac{6}{\pi} .$$

The fact that $\mu_n > \frac{6}{\pi}$ for each n was established in Theorem 3.7(c), and so $\mu = \frac{6}{\pi}$.

Though not stated as such, a consequence of the next result is that the Froude number F is a weakly continuous function from S, with the topology of $\mathbb{R}\times L_2(0,\pi)$, into $\mathbb{R}$.

(F) We now prove that

$$\frac{1}{\mu_n}+\int_0^{\pi}f(t)\sin\theta_n(t)\,dt \to \frac{1}{\mu}+\int_0^{\pi}f(t)\sin\theta(t)\,dt , \tag{3.49}$$

and $f\theta_n \to f\theta$ in $L_1(0,\pi)$ as $n\to\infty$.

Let r, ρ_n, and ρ be as defined in (3.32), (3.40) and (3.43). Let

$$p_n = \frac{1}{\mu_n} + \int_0^\pi f(t)\sin\theta_n(t)\,dt, \quad \text{and} \quad p = \frac{1}{\mu} + \int_0^\pi f(t)\sin\theta(t)\,dt \quad .$$

Then it is the result of Theorem 3.7(c) that $\{p_n\}$ is bounded in $[\frac{\pi}{6},\infty)$ and $p \geq \frac{\pi}{6}$. It is sufficient to prove that every convergent subsequence of $\{p_n\}$ converges to p, and so with no loss of generality, we may assume that $p_n \to q \geq \frac{\pi}{6}$ as $n \to \infty$. If $q = \frac{\pi}{6}$, then Fatou's lemma gives

$$\frac{\pi}{6} \leq \quad p = \frac{1}{\mu} + \int_0^\pi f(t)\sin\theta(t)\,dt$$

$$\leq \lim_{n\to\infty} \left(\frac{1}{\mu_n} + \int_0^\pi f(t)\sin\theta_n(t)\,dt\right) = \lim p_n = q = \frac{\pi}{6} \quad ,$$

and so $p_n \to p$. That $f\theta_n \to f\theta$ in $L_1(0,\pi)$ follows by the arguments below.

Now suppose that $q > \frac{\pi}{6}$, and put

$$\alpha_n = \int_0^\pi r(t)(\theta_n(t) - \theta(t))\,dt, \quad \text{and} \quad \alpha_n' = \int_0^\pi (\rho_n(t) - \rho(t))\,dt \quad .$$

Both α_n and α_n' converge to zero as $n \to \infty$. Then

$$\int_0^\pi f(t)\theta_n(t)\,dt - \int_0^\pi f(t)\theta(t)\,dt$$

$$= \alpha_n + \frac{1}{6}\int_0^\pi \frac{t\, f(t)\sin\theta_n(t)}{\frac{1}{\mu_n} + \int_0^t f(w)\sin\theta_n(w)\,dw}\,dt - \frac{1}{6}\int_0^\pi \frac{t\, f(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f(w)\sin\theta(w)\,dw}\,dt$$

$$= \alpha_n + \frac{1}{2}\alpha_n' + \frac{\pi}{6}(\ln p_n - \ln p)$$

$$= \alpha_n + \frac{1}{2}\alpha_n' + \frac{\pi}{6q_n}(p_n - p) \quad ,$$

by the mean value theorem, where q_n lies between p and p_n. Since, by assumption $p_n \to q > \frac{\pi}{6}$, it follows that if $k_n = \frac{\pi}{6q_n}$, then $k_n \to k < 1$. Therefore

$$\int_0^\pi f(t)(\theta_n(t) - k_n \sin\theta_n(t))dt =$$

$$k_n\left(\frac{1}{\mu_n} - \frac{1}{\mu}\right) + \alpha_n + \frac{1}{2}\alpha'_n + \int_0^\pi f(t)(\theta(t) - k_n \sin\theta(t))dt \quad .$$

Hence

$$\lim_{n\to\infty} \int_0^\pi f(t)(\theta_n(t) - k_n \sin\theta_n(t))dt = \int_0^\pi f(t)(\theta(t) - k\sin\theta(t))dt \quad . \tag{3.50}$$

It follows from Fatou's lemma and (3.50) that

$$2\int_0^\pi f(t)(\theta(t) - k\sin\theta(t))dt$$

$$\leq \liminf_{n\to\infty} \int_0^\pi \{f(t)(\theta(t) - k\sin\theta(t) + \theta_n(t) - k_n \sin\theta_n(t))$$

$$- f(t)\left|(1-k)\theta(t) - (1-k_n)\theta_n(t)\right|\}dt$$

$$= 2\int_0^\pi f(t)(\theta(t) - k\sin\theta(t))dt - \limsup_{n\to\infty} \int_0^\pi f(t)\left|(1-k)\theta(t) - \right.$$

$$\left.(1-k_n)\theta_n(t)\right|dt \quad .$$

Hence

$$\limsup_{n\to\infty} (1-k)\int_0^\pi f(t)\left|\theta(t) - \theta_n(t)\right| \leq 0 \quad ,$$

whence $f\theta_n \to f\theta$ in $L_1(0,\pi)$ as $n \to \infty$.

(G) Finally we compete the proof of this theorem by showing that $(\mu_n, \theta_n) \to (\mu, \theta)$ in $\mathbb{R} \times C_0[0,\pi]$.

Let $\varepsilon > 0$ and choose L_ε such that

$$\frac{\mu_n \pi}{12} \int_{L_\varepsilon}^{\infty} (\ell n \left|\frac{1+t}{1-t}\right|) \frac{1}{t}\, dt \leq \frac{\varepsilon}{2} \quad ,$$

for all positive integers n. Then the argument for (3.45) and (3.46) with θ replaced by θ_n yields

$$\theta_n(s) \leq M\{ \int_{\pi - L_\varepsilon(\pi-s)}^{\pi} \frac{\theta_n(t)}{(\pi-t)}\, dt \}^{1/2} + \frac{\varepsilon}{2} + O(\pi-s) \quad ,$$

where the last term on the right hand side is uniformly $O(\pi-s)$, independently of n, and M is independent of n. Since $f\theta_n \to f\theta$ in $L_1(0,\pi)$, it follows that

$$\theta_n(s) \leq M\{ \int_{\pi - L_\varepsilon(\pi-s)}^{\pi} \frac{\theta(t)}{(\pi-t)}\, dt \}^{1/2} + \varepsilon + O(\pi-s)$$

for all s such that $(\pi-s) < \pi/L_\varepsilon$, and for all n sufficiently large. Hence there exists $\delta > 0$ (depending on ε) such that,

$$\limsup_{n\to\infty} \{ \max_{s\in[\pi-\delta,\pi]} \theta_n(s)\} < 2\varepsilon \quad .$$

Combining this result with (B), it follows that $\{\theta_n\}$ is a Cauchy sequence in $C_0[0,\pi]$, and so $\theta_n \to \theta$ uniformly on $[0,\pi]$. q.e.d.

The global existence result for solitary water waves is now immediate, and is described in the next two theorems.

THEOREM 3.9. *Let* $\mathcal{C}$ *denote the maximal connected subset of* $\mathcal{S}$ *which contains* $(\frac{6}{\pi},0)$. *Then*

(a) $\mathcal{C}$ *is closed and unbounded in* $\mathbb{R} \times C_0[0,\pi]$.

(b) $\{\mu : (\mu,\theta) \in \mathcal{C}\} = [\frac{6}{\pi},\infty)$.

(c) $0 < \theta(s) < \frac{\pi}{2}$ *if* $s \in (0,\pi)$ *and* $(\mu,\theta) \in \mathcal{C}\backslash\{(\frac{6}{\pi},0)\}$.

(d) $\theta \in C^{1,\alpha}[0,\pi-\delta]$ *for each* $\delta \in (0,\pi)$ *and for each* $\alpha \in (0,1)$.

(e) *There exists an absolute constant* M *such that if* $(\mu,\theta) \in \mathcal{C}$, *then*

$$\frac{\pi}{6} \leq \frac{\pi}{6} F^2 = \frac{1}{\mu} + \int_0^\pi f(w) \sin \theta(w) dw \leq M \quad .$$

(f) *If* $0 \leq \beta < \frac{\pi}{6}$, *then there exists* $(\mu,\theta) \in \mathcal{C}$ *such that* $|\theta|_{C_0[0,\pi]} = \beta$.

Remarks. (i) The result of (e) is improved in section 4.5 where it is shown that $F^2 > 1$ if $\theta \neq 0$.

(ii) Using the methods of [33], it is clear that the result in (f) can be improved to say that there also exist $(\mu,\theta) \in \mathcal{C}$ with $|\theta|_{C_0[0,\pi]} > \pi/6$.

Proof of Theorem 3.9. By Theorem 3.8 and Theorem A6, it suffices to show that if U is a bounded open set in $\mathbb{R} \times C_0[0,\pi]$ and $(\frac{6}{\pi},0) \in U$, then $\partial U \cap \mathcal{S} \neq \phi$.

Let U be such an open set. The result of Corollary 3.6 is that, for all n sufficiently large, $\mathcal{C}_n \cap \partial U \neq \phi$. Since U is bounded, it follows that there exists a sequence $\{(\mu^{(n)},\theta^{(n)})\} \subset (0,\infty) \times (K_0\backslash\{0\})$ and an increasing sequence $\{k(n)\}$ of positive integers such that

$$(\mu^{(n)},\theta^{(n)}) \in \mathcal{C}_{k(n)} \cap \partial U \quad , \tag{3.51}$$

$$\mu_n \to \mu \quad , \tag{3.52}$$

$$\theta^{(n)} \rightharpoonup \theta , \tag{3.53}$$

and

$$\sin \theta^{(n)} \rightharpoonup \sigma , \tag{3.54}$$

where θ and σ are in $L_2(0,\pi)$. It follows by the method used in the proof of Theorem 3.8 that $\{(\mu^{(n)},\theta^{(n)})\}$ converges in $\mathbb{R} \times C_0[0,\pi]$ to an element of S. It suffices here to outline the various stages of the proof.

(A') If $\{(\mu^{(n)},\theta^{(n)})\}$, μ, θ and σ are as in (3.51) - (3.54), define $\rho^{(n)}$ as in (3.40) with f and θ_n replaced by $f_{k(n)}$ and $\theta^{(n)}$, respectively. Then $\rho^{(n)} \to \rho$ in $L_2(-\pi,\pi)$, where ρ is given by (3.43), using μ and σ from (3.52) and (3.54). Since $\theta^{(n)} = \underset{\sim}{C}\rho^{(n)}$, there results that $\theta^{(n)} \to \theta$ in $L_2(-\pi,\pi)$ as $n \to \infty$. Moreover $\sigma = \sin \theta$.

(B') As in (B), one can show that $\theta^{(n)} \to \theta$ in $C[0,\pi-\delta]$ for any $\delta \in (0,\pi)$.

(C') The proof that $\{\frac{1}{\mu^{(n)}} + \int_0^\pi f_{k(n)}(t) \sin \theta^{(n)}(t)dt\}$ is bounded follows from the fact that $\{\mu^{(n)}\}$ is bounded in $(\frac{6}{\pi},\infty)$, by the method used in the proof of Theorem 3.7(c). From Fatou's lemma, therefore, $\frac{1}{\mu} + \int_0^\pi f(t)\sin \theta(t)dt < \infty$.

(D') It is now immediate from the proof of (D) that (μ,θ) satisfies (1.26) and $\theta \in K_0$.

(E') To prove that $(\mu,\theta) \in S$, it suffices, as in (E), to show that if $\theta = 0$, then $\mu = \frac{6}{\pi}$. Let $\{g_m\}$, $\{(\gamma_m,\psi_m)\}$, $\{B_{m,n}\}$ be defined as in (E). Then since $(\mu^{(n)},\theta^{(n)}) \in C_{k(n)}$, and $k(n) \geq n$ for all n,

$$\gamma_m \int_0^\pi \psi_m(s)g_m(s)\theta^{(n)}(s)ds = \int_0^\pi \frac{\psi_m(t)f_{k(n)}(t)\sin \theta^{(n)}(t)}{\frac{1}{\mu^{(n)}} + \int_0^t f_{k(n)}(w)\sin \theta^{(n)}(w)dw} dt$$

$$\geq \int_0^\pi \frac{\psi_m(t)g_m(t)\sin \theta^{(n)}(t)}{\frac{1}{\mu^{(n)}} + \int_0^t g_m(w)\sin \theta^{(n)}(w)dw} dt ,$$

if $m \leq n$,

$$\geq B_{m,n}\left\{\frac{\int_0^\pi \psi_m(t) g_m(t)\theta^{(n)}(t)dt}{\frac{1}{\mu^{(n)}} + \int_0^\pi g_m(w)\sin\theta^{(n)}(w)dw}\right\} .$$

Therefore if $m \leq n$,

$$\gamma_m\left(\frac{1}{\mu^{(n)}} + \int_0^\pi g_m(w)\sin\theta^{(n)}(w)dw\right) \geq B_{m,n} ,$$

and the result now follows as before.

(F') We now indicate the proof that $p^{(n)} = \frac{1}{\mu^{(n)}} + \int_0^\pi f_{k(n)}(t)\sin\theta^{(n)}(t)dt$ $\to \frac{1}{\mu} + \int_0^\pi f(t)\sin\theta(t)dt = p$, and that $f_{n(k)}\theta^{(n)} \to f\theta$ in $L_1(0,\pi)$ as $n \to \infty$. As in (F) we show that every convergent subsequence of $\{p^{(n)}\}$ converges to p. So let $p^{(n)} \to q$. By Fatou's lemma, (D') and Theorem 3.7(c), there results that $q \geq \frac{\pi}{6}$. If $q = \frac{\pi}{6}$, then the result is immediate, and if $q > \frac{\pi}{6}$, the method of (F) gives the required conclusion.

(G') The convergence of $\{(\mu^{(n)},\theta^{(n)})\}$ to (μ,θ) in $\mathbb{R} \times C_0[0,\pi]$ follows by the method of (G). This completes the proof of (a).

Parts (b) - (e) are a direct corollary of (a) and Theorem 3.7.

(f) By Theorem 5.1, there exists a sequence $\{(\mu_n,\theta_n)\} \subset \mathcal{C}$ such that $\theta_n \to \theta^*$ in $C[\delta,\pi]$ for each $\delta \in (0,\pi)$, where θ^* satisfies (5.1). By Theorem 5.2, $\limsup_{s\to 0+} \theta^*(s) \geq \frac{\pi}{6}$, whence $\sup\{\theta(s) : s \in [0,\pi], (\mu,\theta) \in \mathcal{C}\} \geq \pi/6$. Since $\mathcal{C}$ is a connected set in $\mathbb{R} \times C_0[0,\pi]$, the result follows. q.e.d.

Our next result concerns the global existence of solitary waves, when the bifurcation parameter is the dimensionless Froude number. Let

$$\mathcal{D} = \{(F^2,\theta) : F^2 = \frac{6}{\pi}\left(\frac{1}{\mu} + \int_0^\pi f(t)\sin\theta(t)dt\right) \text{ where } (\mu,\theta) \in \mathcal{C}\}.$$

THEOREM 3.10. (a) The set $\mathcal{D}$ is closed, bounded and connected in $\mathbb{R} \times C_0[0,\pi]$, but is not compact.

(b) $\{F^2 : (F^2,\theta) \in \mathcal{D}\} = [1,M]$ for some finite constant M, and $\{F^2 : (F^2,\theta) \in \mathcal{D}\setminus\{(1,0)\}\} = (1,M)$.

Proof. That $\mathcal{D}$ is bounded and connected is a result of Theorem 3.9(c) and (e), while its connectedness and closedness follows from the proof of Theorem 3.8. Now Theorems 5.1 and 5.2(f) combine to say that there exists a sequence $\{(\mu_n,\theta_n)\} \subset C$ with $\mu_n \to \infty$ such that $\{\theta_n\}$ is not Cauchy in $C_0[0,\pi]$. Therefore $\mathcal{D}$ is not compact.

(b) The first part of (b) has already been established in Theorem 3.9(e), and the second part is proved by a long and delicate analysis in Theorem 4.12.

q.e.d.

Remark. In Theorem 4.6 a formula is given for calculating the profile $\{(x, H_{\mu,\theta}(x) : x \in \mathbb{R}\}$ of a solitary wave corresponding to a solution (μ,θ) of (1.26). It is a corollary of Theorems 3.10 and 4.6 that the set $\{H_{\mu,\theta}: (\mu,\theta) \in C\}$ is a closed, connected subset of $C(\mathbb{R})$; that it is compact in $C(\mathbb{R})$ follows from Theorem 5.1.

4. PROPERTIES OF SOLITARY WAVES

4.1. Introductory remarks

In section 3.3 we constructed non-trivial elements of S as the limit of solutions $(\mu^{(n)},\theta^{(n)}) \in (0,\infty) \times K_0$ of (3.10). It is conceivable that the use of other methods might lead to solutions of (1.26) in a wider class $\tilde{S}$, where

$$\tilde{S} = \{(\mu,\theta) \in (0,\infty) \times C[0,\pi) : (\mu,\theta) \text{ satisfies (1.26) on } [0,\pi) \text{ and}$$
$$0 < \theta(s) \leq \pi \text{ on } (0,\pi)\} \cup \{(\tfrac{6}{\pi},0)\} .$$

Clearly $S \subset \tilde{S}$; we shall show in section 4.2 that if $(\mu,\theta) \in \tilde{S}$, then $\theta f \in L_1(0,\pi)$ and $\theta \in K_0$, whence $\tilde{S} = S$. In section 4.3 we show that if $(\mu,\theta) \in S$, then θ is analytic on $(-\pi,\pi)$. With this result and Theorem 3.7(b), the proof of Theorem 1.1 will be complete. The analyticity of θ ensures that the solitary wave $\Gamma = \{(x,y) : y = H(x),\ x \in \mathbb{R}\}$ has H analytic on $\mathbb{R}$.

In section 4.4 we show how to recover the solitary wave profile from solutions of (1.26). In section 4.5, we consider an important facet of the solitary wave problem namely, the possible values of the Froude number F. Recall from Theorem 3.7(c) that if $(\mu,\theta) \in S$, then $F \geq 1$. If $F > 1$, we show that $\theta(s)$ goes to zero algebraically at $s = \pi$ (Theorem 4.7). In Theorem 4.8, we give necessary and sufficient conditions on θ and on H to ensure that $F > 1$. By a complicated argument in Theorem 4.12, we show that $F > 1$ for non-trivial elements of S. Finally, we prove in section 4.6 that there does not exist a non-trivial solution of an equation analogous to (1.26) which would correspond to a solitary wave of depression.

4.2. $S = \tilde{S}$

THEOREM 4.1. *If* $(\mu,\theta) \in \tilde{S}$, *then* $\theta f \in L_1(0,\pi)$.

Proof. The use of Theorem 2.5(f) yields

$$\theta(s) \le \frac{1}{3\pi}\int_0^{\pi} \ln\left|\frac{2\pi-s-t}{s-t}\right| \frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f(w)\sin\theta(w)dw}\,dt + \text{const.}(\pi-s)$$

and so for each $\varepsilon \in (0,\pi)$,

$$\int_0^{\pi-\varepsilon} (\pi-s)^{-1}\theta(s)ds \le \int_0^{\pi} Y(t;\varepsilon)\frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f\sin\theta}\,dt + \text{const.} \quad , \qquad (4.1)$$

where

$$Y(t;\varepsilon) = \frac{1}{3\pi}\int_0^{\pi-\varepsilon} (\pi-s)^{-1}\ln\left|\frac{2\pi-s-t}{s-t}\right|ds$$

$$= \frac{1}{3\pi}\int_{\varepsilon/(\pi-t)}^{\pi/(\pi-t)} x^{-1}\ln\left|\frac{1+x}{1-x}\right|dx, \quad t \in (0,\pi) \quad .$$

Now

$$Y(t;\varepsilon) \le \frac{1}{3\pi}\int_0^{\infty} x^{-1}\ln\left|\frac{1+x}{1-x}\right|dx = \frac{\pi}{6} \quad ,$$

and if $t \in (\pi-\varepsilon,\pi)$, then $\varepsilon > (\pi-t)$, and so

$$Y(t;\varepsilon) \le \text{const. } \varepsilon^{-1}(\pi-t) \quad ,$$

where the constant is independent of ε. The use of these estimates in (4.1) yield

$$\int_0^{\pi-\varepsilon} (\pi-s)^{-1}\theta(s)ds \le \frac{\pi}{6}\int_0^{\pi-\varepsilon} \frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f\sin\theta}\,dt + \text{const. } \varepsilon^{-1}\int_{\pi-\varepsilon}^{\pi} \theta(t)dt + \text{const.}$$

$$\le \frac{\pi}{6}\ln\left(\frac{1}{\mu}+\int_0^{\pi-\varepsilon} f\sin\theta\right) + \text{const.} \quad .$$

Since $f(s)\sin\theta(s) \le \frac{\pi}{2}(\pi-s)^{-1}\theta(s)$, it follows that

$$\int_0^{\pi-\varepsilon} (\pi-s)^{-1}\theta(s)ds \le \frac{\pi}{6}\ln\left(\frac{1}{\mu}+\frac{\pi}{2}\int_0^{\pi-\varepsilon} (\pi-s)^{-1}\theta(s)ds\right) + \text{const.} \quad ,$$

whence $|\theta f|_{L_1(0,\pi-\varepsilon)}$ is bounded independently of ε. q.e.d.

The proof that $\theta(0) = 0$ is elementary, and so the proof that $S = \tilde{S}$ will follow from the following theorem.

THEOREM 4.2. _If_ $(\mu,\theta) \in \tilde{S}$, _then_ $\theta(s) \to 0$ _as_ $s \to \pi$.

Proof. Since $\theta f \in L_1(0,\pi)$, the proof of this theorem follows as in (D) of the proof of Theorem 3.8. q.e.d.

4.3. _Analyticity of_ θ _and the solitary wave profile_

The argument for (3.40) - (3.42) shows that

$$\theta = \underset{\sim}{C}\tau \quad \text{and} \quad \underset{\sim}{C}\theta = -\tau + a_0 \quad ,$$

where

$$\tau(t) = -\frac{1}{3}\ln\left(\frac{1}{\mu} + \int_0^t f(w)\sin\theta(w)dw\right) + \frac{1}{3}\ln\left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)dw\right), \quad t \in (-\pi,\pi),$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tau(t)dt \quad ,$$

and $\underset{\sim}{C}$ denotes 'conjugate'. Theorem 3.7(a) ensures that $\theta \in C^{1,\alpha}[a,b]$ for all closed intervals $[a,b] \subset (-\pi,\pi)$, and the form of τ ensures that $\tau \in C^{2,\alpha}[a,b]$ for all $\alpha \in (0,1)$. Suppose that

$$\theta(s) = \sum_{k=1}^{\infty} a_k \sin ks, \text{ for } s \in (-\pi,\pi) \quad ,$$

and define the analytic function

$$\tilde{\omega}(re^{it}) = a_0 + \sum_{k=1}^{\infty} a_k r^k e^{ikt}, \quad r \in [0,1), \quad t \in (-\pi,\pi) \quad ,$$

so that $\tilde{\omega} = \tilde{\tau} + i\tilde{\theta}$, where $\tilde{\tau}(e^{it}) = \tau(t)$ and $\tilde{\theta}(e^{it}) = \theta(t)$ for $t \in (-\pi,\pi)$. Arguing as in the proof of Theorem 3.3(d), we have

$$\frac{\partial\tilde{\theta}}{\partial r}(e^{it}) = -\frac{\partial\tilde{\tau}}{\partial t}(e^{it}) = \frac{1}{3}f(t)\sin\theta(t)\left(\frac{1}{\mu} + \int_0^t f(w)\sin\theta(w)dw\right)^{-1}$$

$$= \frac{1}{3}\left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)dw\right)^{-1} f(t)\sin\theta(t)\exp(3\tau(t)),\ t \in (-\pi,\pi) \quad . \tag{4.2}$$

Equation (4.2) is of a form examined by Lewy [28], and his theorem ensures that the harmonic function $\tilde{\theta}(\zeta)$ may be extended to be real-analytic in an open neighborhood of $\zeta = e^{it}$ for each $t \in (-\pi,\pi)$. In particular, θ is analytic on $(-\pi,\pi)$, and so we have proved

THEOREM 4.3. If $(\mu,\theta) \in S$, then θ is analytic on $(-\pi,\pi)$.

Recall from section 1.2 that the solitary wave profile Γ is given by

$$\Gamma = \{(x,y) : y = H(x),\ x \in \mathbb{R}\} \quad .$$

COROLLARY 4.4. The function H is analytic on $\mathbb{R}$.

Proof. With the notation of section 1.2, we have

$$\frac{dH}{dx}(x) = \tan\Theta(\gamma(x)) \quad ,$$

where $\Theta(s) = \theta(2s)$, and γ is the inverse of the function

$$\lambda(s) = \text{Real } m(e^{is}),\ s \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \quad ,$$

given in (1.25). The use of Theorem 4.3 and (1.25) together ensure that γ is analytic on $\mathbb{R}$. q.e.d.

For a non-trivial element $(\mu,\theta) \in S$, we know that $\theta \in C_0[0,\pi]$ and that θ is analytic on $[0,\pi)$. The following theorem and the mean-value theorem together show that

$$\liminf_{s\to\pi} \frac{d\theta}{ds}(s) = -\infty \quad .$$

THEOREM 4.5. For each non-trivial $(\mu,\theta) \in S$, there exists a positive constant C such that

$$\theta(s) \geq C(\pi-s)\ln(\frac{\pi}{\pi-s})$$

for all $s \in (0,\pi)$.

Proof. The use of Theorem 2.5(c) in (1.26) ensures that $\theta(s) \geq \delta \sin s$ on $(0,\pi)$ for some $\delta > 0$. If we combine this estimate with Theorem 2.5(f), the result is

$$\begin{aligned}\theta(s) &\geq \text{const.}(\pi-s) + \text{const.}\int_{\pi/2}^{\pi} \ln\left|\frac{2\pi-s-t}{s-t}\right|dt \\ &\geq \text{const.}(\pi-s)\ln(\frac{\pi}{\pi-s})\end{aligned}$$

for all $s \in (0,\pi)$. q.e.d.

It follows by induction that $\theta(s) \geq \text{const.}(\pi-s)(\ln(\frac{\pi}{\pi-s}))^m$, $s \in (0,\pi)$, for each positive integer m, where the constant depends on m, μ, and θ.

4.4. The solitary wave profile

For each $(\mu,\theta) \in S$, the physical flow parameters h and c for the corresponding solitary wave profile $\Gamma = \{(x,y) : y = H(x),\ x \in \mathbb{R}\}$ are determined as in (1.18a) by the relationship

$$\frac{c^2}{gh} = F^2 = \frac{6}{\pi}(\frac{1}{\mu} + \int_0^{\pi} f(w)\sin\theta(w)dw) \quad .$$

For h and c so chosen, the following theorem shows how to calculate the function H from (μ,θ).

THEOREM 4.6. For each $(\mu,\theta) \in S$, the corresponding solitary wave profile is given by

$$H(x) = h + \frac{h}{2}(\frac{6}{\pi}F)^{2/3}\{(\frac{1}{\mu} + \int_0^{\pi} f(w)\sin\theta(w)dw)^{2/3} - (\frac{1}{\mu} + \int_0^{2\lambda^{-1}(x)} f(w)\sin\theta(w)dw)^{2/3}\}, \tag{4.3}$$

where

$$\lambda(s) = -\frac{h}{3}(\frac{6}{\pi}F)^{2/3}\int_0^{2s} f(t)\cos\theta(t)(\frac{1}{\mu} + \int_0^{t} f(w)\sin\theta(w)dw)^{-1/3}dt,\ s \in (-\frac{\pi}{2},\frac{\pi}{2}) \quad . \tag{4.4}$$

The amplitude of the wave is given by

$$H(0) - H(-\infty) = H(0) - h = \frac{h}{2}\{F^2 - (\frac{6F}{\pi\mu})^{2/3}\} .$$

Proof. With the notation of the proof of Corollary 4.4

$$H(x) - H(0) = \int_0^x \tan \Theta(\gamma(u))du = \int_0^{\lambda^{-1}(x)} \tan \Theta(s)\frac{d\lambda}{ds}(s)ds$$

$$= -\frac{2h}{\pi}(\frac{1}{\mu} + \int_0^{\pi/2} \sec w \sin \Theta(w)dw)^{1/3} \int_0^{\lambda^{-1}(x)} \sec t \sin \Theta(t)(\frac{1}{\mu} + \int_0^t \sec w \sin \Theta(w)dw)^{-1/3}dt$$

$$= -\frac{h}{3}(\frac{6}{\pi} F)^{2/3} \int_0^{2\lambda^{-1}(x)} f(t)\sin \theta(t)(\frac{1}{\mu} + \int_0^t f(w)\sin \theta(w)dw)^{-1/3} dt , \tag{4.5}$$

where we have used the relation $\Theta(w) = \theta(2w)$ and the representation for λ and $\frac{d\lambda}{ds}$ given in (1.25). It follows that

$$H(x) - H(0) = -\frac{h}{2}(\frac{6}{\pi} F)^{2/3} \{(\frac{1}{\mu} + \int_0^{2\lambda^{-1}(x)} f(w)\sin \theta(w)dw)^{2/3} - \mu^{-2/3}\} ,$$

and letting $x \to -\infty$ gives $2\lambda^{-1}(x) \to \pi$, whence

$$H(-\infty) - H(0) = -\frac{h}{2}(\frac{6}{\pi} F)^{2/3} \{(\frac{1}{\mu} + \int_0^{\pi} f(w)\sin \theta(w)dw)^{2/3} - \mu^{-2/3}\} .$$

q.e.d.

4.5. The Froude number F

Recall from Theorem 3.7(c) that $F^2 \geq 1$ for any $(\mu,\theta) \in S$. Various physical and numerical evidence points to a strict inequality here for all non-trivial solitary waves, and we establish this result in Theorem 4.12. The significance of this observation lies in the fact that if $F^2 > 1$, then a fairly precise description of $\theta(s)$ near $s = \pi$ may be given, from which the rate of asymptotic decay of the wave may be deduced. This gives a rigorous justification for some of the assumptions made in [20].

THEOREM 4.7. Let $(\mu,\theta) \in S$ and assume that

$$F^2 = \frac{6}{\pi}\left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)dw\right) > 1 \quad .$$

Let $\tilde{\alpha} \in (0,1)$ be the unique number such that

$$\frac{2}{\pi\tilde{\alpha}} \tan\left(\frac{\pi\tilde{\alpha}}{2}\right) = F^2 \quad .$$

Then

(a) $(\pi-s)^{-\alpha}\theta(s) \to 0$ as $s \to \pi$ for all $\alpha \in [0,\tilde{\alpha})$.

(b) If $\alpha \in (\tilde{\alpha},1)$ and $\liminf_{s\to\pi} (\pi-s)^{-\alpha}\theta(s) > 0$, then $(\pi-s)^{-\alpha}\theta(s) \to \infty$ as $s \to \pi$.

(c) If $H(x)$ and h are as in Theorem 4.6, then

$$\lim_{|x|\to\infty} \exp\left(\frac{\pi\alpha|x|}{2h}\right)(H(x) - h) = 0 \quad \forall\, \alpha \in [0,\tilde{\alpha}) \quad .$$

Proof. (a) Let $\alpha \in (0,\tilde{\alpha})$ be fixed and choose $\varepsilon > 0$ such that

$$\varepsilon < 3\alpha \cot\left(\frac{\pi\alpha}{2}\right) - \frac{6}{\pi} F^{-2} \quad . \tag{4.6}$$

Now choose $\delta > 0$ such that

$$0 < \left(\frac{1}{\mu} + \int_0^t f(w)\sin\theta(w)dw\right)^{-1} - \frac{6}{\pi} F^{-2} \leq \varepsilon \quad \text{for} \quad t \in (\pi-\delta,\pi) \quad . \tag{4.7}$$

We have

$$\theta(s) = \frac{2}{3}\int_0^\pi G(s,t)\frac{\sin\theta(t)}{(\pi-t)\left(\frac{1}{\mu} + \int_0^t f\sin\theta\right)}dt + \frac{2}{3}\int_0^\pi G(s,t)\frac{\sin\theta(t)\{f(t)-(\pi-t)^{-1}\}}{\frac{1}{\mu} + \int_0^t f\sin\theta}dt$$

$$= \frac{2}{3}\int_0^\pi G(s,t)\frac{\sin\theta(t)}{(\pi-t)\left(\frac{1}{\mu} + \int_0^t f\sin\theta\right)}dt + A_1(s) \quad ,$$

where $A_1 \in C[0,\pi]$ and $A_1(s) = O(\pi-s)$. The use of Theorem 2.5(f) yields

$$\frac{2}{3}\int_0^{\pi} G(s,t)\frac{\sin\theta(t)}{(\pi-t)(\frac{1}{\mu}+\int_0^t f\sin\theta)}\,dt = \frac{1}{3\pi}\int_{\pi-\delta}^{\pi} \ln\left|\frac{2\pi-s-t}{s-t}\right| \frac{\sin\theta(t)}{(\pi-t)(\frac{1}{\mu}+\int_0^t f\sin\theta)}\,dt$$

$$+\frac{1}{3\pi}\int_0^{\pi-\delta} \ln\left|\frac{2\pi-s-t}{s-t}\right| \frac{\sin\theta(t)}{(\pi-t)(\frac{1}{\mu}+\int_0^t f\sin\theta)}\,dt + \frac{2}{3}\int_0^{\pi} h(s,t)\frac{\sin\theta(t)}{(\pi-t)(\frac{1}{\mu}+\int_0^t f\sin\theta)}\,dt \quad , \tag{4.8}$$

and the last two terms on the right of (4.8) lie in $C[0,\pi]$ and are $O(\pi-s)$. Hence, we have

$$\theta(s) = \frac{1}{3\pi}\int_{\pi-\delta}^{\pi} \ln\left|\frac{2\pi-s-t}{s-t}\right| \frac{\sin\theta(t)}{(\pi-t)(\frac{1}{\mu}+\int_0^t f\sin\theta)}\,dt + B(s), \quad s\in(0,\pi) \quad , \tag{4.9}$$

where $B\in C[0,\pi]$ and $B(s) = O(\pi-s)$. For suitable functions ψ, define

$$N\psi(s) = \frac{1}{3\pi}\int_{\pi-\delta}^{\pi} \ln\left|\frac{2\pi-s-t}{s-t}\right| \frac{\sin\psi(t)}{(\pi-t)(\frac{1}{\mu}+\int_0^t f\sin\theta)}\,dt \quad ,$$

and set $T\psi(s) = N\psi(s) + B(s)$. Note that the function θ satisfies $\theta(s) = T\theta(s)$ for $s\in[\pi-\delta,\pi]$ (indeed, for $s\in[0,\pi]$).

Define the Banach space

$$X_0 = \{f\in C[\pi-\delta,\pi] : f(\pi) = 0\}$$

with $\|f\|_0 = |f|_{X_0} = \sup_{s\in[\pi-\delta,\pi]} |f(s)|$. For each $\alpha\in(0,1)$, define the Banach space

$$X_\alpha = \{f\in C[\pi-\delta,\pi] : \text{the mapping } s\mapsto(\pi-s)^{-\alpha}f(s) \text{ is in } X_0\}$$

with $\|f\|_\alpha = |f|_{X_\alpha} = \sup_{s\in[\pi-\delta,\pi]} (\pi-s)^{-\alpha}|f(s)|$. We now show that T maps X_α into X_α. If $\psi\in X$, then the use of (4.7) yields

$$|T\psi(s)| \le \frac{1}{3\pi}(\frac{6}{\pi} F^{-2} + \varepsilon)\int_{\pi-\delta}^{\pi} \ell n\left|\frac{2\pi-s-t}{s-t}\right| \left|\frac{\sin \psi(t)}{(\pi-t)^{\alpha}}\right| (\pi-t)^{\alpha-1}dt + |B(s)|$$

$$\le \frac{1}{3\pi}(\frac{6}{\pi} F^{-2} + \varepsilon) ||\psi||_{\alpha} (\pi-s)^{\alpha}\int_{0}^{\delta/(\pi-s)} x^{\alpha-1}\ell n\left|\frac{1+x}{1-x}\right|dx + |B(s)|$$

$$\le \frac{1}{3\pi}(\frac{6}{\pi} F^{-2} + \varepsilon) ||\psi||_{\alpha} (\pi-s)^{\alpha}\int_{0}^{\infty} x^{\alpha-1}\ell n\left|\frac{1+x}{1-x}\right|dx + |B(s)| \quad . \tag{4.10}$$

Hence

$$\sup_{s\in(\pi-\delta,\pi)} (\pi-s)^{-\alpha}|T\psi(s)| < \infty \quad ,$$

and to show that $T\psi \in X_{\alpha}$, we must prove that $(\pi-s)^{-\alpha}T\psi(s) \to 0$ as $s \to \pi$. Let $\tilde{\varepsilon} > 0$ and choose $\tilde{\delta} \in (0,\delta)$ such that $|(\pi-s)^{-\alpha}\psi(s)| \le \tilde{\varepsilon}$ for all $s \in (\pi-\tilde{\delta},\pi)$. The argument above gives

$$|T\psi(s)| \le \frac{1}{3\pi}(\frac{6}{\pi} F^{-2} + \varepsilon)\tilde{\varepsilon}(\pi-s)^{\alpha}\int_{0}^{\tilde{\delta}/(\pi-s)} x^{\alpha-1}\ell n\left|\frac{1+x}{1-x}\right|dx$$

$$+ \frac{1}{3\pi}(\frac{6}{\pi} F^{-2} + \varepsilon) ||\psi||_{\alpha} (\pi-s)^{\alpha}\int_{\tilde{\delta}/(\pi-s)}^{\infty} x^{\alpha-1}\ell n\left|\frac{1+x}{1-x}\right|dx + |B(s)| \quad ,$$

and so

$$\limsup_{s\to\pi} (\pi-s)^{-\alpha}|T\psi(s)| \le \frac{1}{3\pi}(\frac{6}{\pi} F^{-2} + \varepsilon)\tilde{\varepsilon} \int_{0}^{\infty} x^{\alpha-1}\ell n\left|\frac{1+x}{1-x}\right|dx \quad .$$

Since $\tilde{\varepsilon}$ may be taken arbitrarily small, we have proved that T maps X_{α} into X_{α}.

Since $|\sin x - \sin y| \le |x-y|$ for all $x,y \in \mathbb{R}$, the arguments for (4.10) yield

$$||T\psi_1 - T\psi_2||_{\alpha} \le \frac{1}{3\pi}(\frac{6}{\pi} F^{-2} + \varepsilon)\int_{0}^{\infty} x^{\alpha-1}\ell n\left|\frac{1+x}{1-x}\right|dx \; ||\psi_1 - \psi_2||_{\alpha}$$

$$= \frac{1}{3\alpha} \tan (\frac{\pi\alpha}{2}) (\frac{6}{\pi} F^{-2} + \varepsilon) ||\psi_1 - \psi_2||_{\alpha} = K||\psi_1 - \psi_2||_{\alpha}$$

where $K < 1$ by (4.6), and we have used the formula

$$\int_0^\infty x^{\alpha-1} \ln\left|\frac{1+x}{1-x}\right| dx = \frac{\pi}{\alpha} \tan\left(\frac{\pi\alpha}{2}\right), \quad \alpha \in [0,1) \quad .$$

Hence, the mapping T is a contraction from X_α into X_α, and so there exists a unique $\psi_\alpha \in X_\alpha$ such that

$$\psi_\alpha(s) = T\psi_\alpha(s), \quad s \in [\pi-\delta,\pi] \quad .$$

The same arguments ensure the existence of a unique $\psi_0 \in X_0$ such that $\psi_0(s) = T\psi_0(s)$ for all $s \in [\pi-\delta,\pi]$. Since $X_\alpha \subset X_0$, we have $\psi_\alpha = \psi_0$. However, θ is an element of X_0 and satisfies $\theta = T\theta$, and so $\theta \in X_\alpha$.

(b) Let $\alpha \in (\tilde{\alpha},1)$ and assume that the statement of (b) is false; that is,

$$0 < \liminf_{s\to\pi} (\pi-s)^{-\alpha}\theta(s) < \infty \quad .$$

From (4.9), we have, for $\delta \in (0,\pi)$ sufficiently small,

$$\theta(s) \geq \frac{1}{3\pi}\frac{6}{\pi} F^{-2} \int_{\pi-\delta}^{\pi} \ln\left|\frac{2\pi-s-t}{s-t}\right| \frac{\sin\theta(t)}{\theta(t)} \frac{\theta(t)}{(\pi-t)^\alpha} (\pi-t)^{\alpha-1} dt + B(s)$$

$$\geq \frac{2}{\pi^2} F^{-2}\{[\liminf_{t\to\pi} (\frac{\sin\theta(t)}{\theta(t)} \frac{\theta(t)}{(\pi-t)^\alpha})] - \varepsilon_\delta\} (\pi-s)^\alpha \int_0^{\delta/(\pi-s)} x^{\alpha-1} \ln\left|\frac{1+x}{1-x}\right| dx + B(s)$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$, and $B(s) = O(\pi-s)$. It follows that

$$\liminf_{s\to\pi} (\pi-s)^{-\alpha}\theta(s) \geq \frac{2}{\pi\alpha} \tan\left(\frac{\pi\alpha}{2}\right) F^{-2}\{[\liminf_{t\to\pi} (\frac{\sin\theta(t)}{\theta(t)} \frac{\theta(t)}{(\pi-t)^\alpha})] - \varepsilon_\delta\} \quad .$$

Dividing this relation by the left-hand side and then letting $\delta \to 0$ gives $F^2 \geq \frac{2}{\pi\alpha} \tan\left(\frac{\pi\alpha}{2}\right)$, or, equivalently, $g(\tilde{\alpha}) \geq g(\alpha)$, where $g(x) = \frac{2}{\pi x} \tan\left(\frac{\pi x}{2}\right)$. However, a simple argument shows that $g(x)$ is strictly increasing for $x \in (0,1)$, and since $\tilde{\alpha} < \alpha$, we must have $g(\tilde{\alpha}) < g(\alpha)$. This is a contradiction, and so $\liminf_{s\to\pi} (\pi-s)^{-\alpha}\theta(s) = \infty$.

(c) Since $\theta(\pi) = 0$, it follows from (4.4) and L'Hospital's rule that

$$\lim_{s\to\frac{\pi}{2}} \frac{\lambda(s)}{\ell n(\pi-2s)} = \frac{h}{3}\left(\frac{6}{\pi}F\right)^{2/3}\left(\frac{1}{\mu}+\int_0^{\pi} f(w)\sin\theta(w)\,dw\right)^{-1/3} = \frac{2h}{\pi} .$$

Setting $s = \lambda^{-1}(x)$ yields

$$\lim_{x\to-\infty} \frac{x}{\ell n(\pi-2\lambda^{-1}(x))} = \frac{2h}{\pi} ,$$

and so $2\lambda^{-1}(x) = \pi - \exp(\frac{\pi x}{2h}) + o(\exp(\frac{\pi x}{2h}))$ as $x \to -\infty$. Equation (4.3) gives

$$\frac{H(x)-h}{h} = \frac{1}{2}\left(\frac{6}{\pi}F\right)^{2/3}\left(\frac{\pi}{6}F^2\right)^{2/3}\left\{1-\left(1-\frac{6}{\pi}F^{-2}\int_{2\lambda^{-1}(x)}^{\pi} f(w)\sin\theta(w)\,dw\right)^{2/3}\right\}$$

$$\simeq \frac{1}{2}F^2\left(\frac{2}{3}\frac{6}{\pi}F^{-2}\int_{2\lambda^{-1}(x)}^{\pi} f(w)\sin\theta(w)\,dw\right) \simeq \frac{2}{\pi}\int_{2\lambda^{-1}(x)}^{\pi} (\pi-w)^{-1}\theta(w)\,dw$$

for large negative x; indeed, one can show rigorously that

$$\lim_{x\to-\infty} \left(\frac{H(x)-h}{h}\right)\Big/\left(\int_{\pi-\exp(\frac{\pi x}{2h})}^{\pi} (\pi-w)^{-1}\theta(w)\,dw\right) = \frac{2}{\pi} . \tag{4.11}$$

Part (a) shows that $\theta(t) \le \text{const.}(\pi-t)^{\alpha}$ for all $\alpha \in [0,\tilde{\alpha})$, whence $H(x) - h \le$ const. $\exp(\frac{\pi\alpha x}{2h})$ for large negative x. A similar argument holds for positive x.

q.e.d.

<u>Remarks on</u> $\alpha = \tilde{\alpha}$. Theorem 4.7(c) and other results [25; pp. 425-426] lead one to believe that

$$\lim_{|x|\to\infty} \exp\left(\frac{\pi\tilde{\alpha}|x|}{2h}\right)(H(x)-h)$$

exists. A highly technical argument, quite different from that for Theorem 4.7, shows that

$$\limsup_{s\to\pi} (\pi-s)^{-\tilde{\alpha}} \int_s^{\pi} (\pi-w)^{-1}\theta(w)\,dw < \infty ,$$

and the use of this in (4.11) yields the following improvement of Theorem 4.7(c):

$$\limsup_{|x|\to\infty} \exp\left(\frac{\pi\alpha|x|}{2h}\right)(H(x)-h) < \infty \quad .$$

The following theorem gives various conditions for F^2 to be strictly greater than one; we shall prove in Theorem 4.12 that (b) is satisfied.

THEOREM 4.8. <u>Let</u> $(\mu,\theta) \in S$ <u>be non-trivial.</u> <u>The following conditions are equivalent</u>

(a) $F^2 > 1$;

(b) $\int_0^\pi (\pi-t)^{-1}\left(\int_t^\pi (\pi-s)^{-1}\theta(s)\,ds\right)dt < \infty$;

(c) $\int_{-\infty}^{\infty} (H(x)-h)\,dx < \infty$, <u>and</u>

$$F^2 = 1 + \frac{3}{2h}\,\frac{\int_{-\infty}^{\infty} (H(x)-h)^2 dx}{\int_{-\infty}^{\infty} (H(x)-h)\,dx} > 1 \quad . \tag{4.12}$$

<u>Remark</u>. Condition (c) states that the area between the line $y = h$ and the curve $y = H(x)$ is finite, or, equivalently, that the mass of the solitary wave is finite.

<u>Proof of Theorem</u> 4.8. (a) $\Rightarrow$ (b) If $F^2 > 1$, then Theorem 4.7 ensures that $\theta(s) \le \text{const.}(\pi-s)^\alpha$ for some $\alpha > 0$, and so (b) is immediate.

(b) $\Rightarrow$ (c) For $M > 0$, equations (4.3) and (4.4) yield

$$\int_{-M}^{0} (H(x)-h)\,dx = \frac{h}{2}\left(\frac{6}{\pi}F\right)^{2/3} \int_{2\lambda^{-1}(-M)}^{0} \left\{\left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)\,dw\right)^{2/3} - \left(\frac{1}{\mu} + \int_0^s f(w)\sin\theta(w)\,dw\right)^{2/3}\right\}\frac{d}{ds}\lambda\left(\frac{s}{2}\right)ds$$

$$= \frac{h^2}{6}\left(\frac{6}{\pi}F\right)^{4/3} \int_0^{2\lambda^{-1}(-M)} f(s)\cos\theta(s)\left(\frac{1}{\mu} + \int_0^s f(w)\sin\theta(w)\,dw\right)^{-1/3}\left\{\left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)\,dw\right)^{2/3} - \left(\frac{1}{\mu} + \int_0^s f(w)\sin\theta(w)\,dw\right)^{2/3}\right\}ds$$

$$\le \text{const.} \int_0^{2\lambda^{-1}(-M)} (\pi-s)^{-1}\left(\int_s^\pi (\pi-w)^{-1}\theta(w)\,dw\right)ds \quad ,$$

where $2\lambda^{-1}(-M) \to \pi$ as $M \to \infty$. Hence, if (b) holds, then

$$\int_{-\infty}^{\infty} (H(x)-h)\,dx < \infty \quad . \tag{4.13}$$

The fact that (4.13) implies (4.12) is standard [30], [42], but for the sake of completeness, we now give a brief proof here. The method of proof is similar in certain respects to that in [29].

Let S denote the physical flow domain:

$$S = \{(x,y) : 0 < y < H(x), \quad x \in \mathbb{R}\} \quad ,$$

and define the pressure p in $\bar{S}$ by $p(x,y) = -g(y-h) - \frac{1}{2}(u^2(x,y) + v^2(x,y)) + \frac{c^2}{2}$. Then p is zero on Γ, by (1.3), (1.4) and (1.7), and

$$p(x,y) + u^2(x,y) + g(y-h) - c^2 + v^2(x,y) + p(x,y) + g(y-h) = 0 \tag{4.14}$$

for all $(x,y) \in \bar{S}$.

From (4.14), we have $p_x = -uu_x - vv_x = -\mathrm{div}(u(u,v))$ by the relation $u_x = -v_y$ and $u_y = v_x$. Integration of this expression for p_x over the region $\{(x',y') : x \le x' \le M,\ 0 \le y' \le H(x')\}$ for $x \in (-\infty,\infty)$ and $M > x$, yields

$$\int_0^{H(x)} (p(x,y) + u^2(x,y))\,dy = \int_0^{H(M)} (p(M,y) + u^2(M,y))\,dy \tag{4.15}$$

where we have used the fact that the normal component of (u,v) vanishes on ∂S and $p = 0$ on Γ. The use of (4.14) in the right-hand side of (4.15) yields

$$\int_0^{H(x)} (p(x,y) + u^2(x,y))\,dy = \frac{1}{2}\int_0^{H(M)} \{u^2(M,y) - v^2(M,y) - 2g(y-h) + c^2\}\,dy$$

$$\to \int_0^{h} \{c^2 - g(y-h)\}\,dy, \quad \text{as} \quad M \to \infty$$

by (1.3) and (1.4). Hence for all $x \in \mathbb{R}$,

$$\int_0^{H(x)} (p(x,y) + u^2(x,y))\,dy = \int_0^{h} \{c^2 - g(y-h)\}\,dy$$

and so

$$\int_S (p(x,y) + u^2(x,y) + g(y-h) - c^2)\,dxdy = \int_{-\infty}^{\infty} dx \int_h^{H(x)} (g(y-h) - c^2)\,dy$$

$$= \frac{g}{2}\int_{-\infty}^{\infty} (H(x)-h)^2 dx - c^2 \int_{-\infty}^{\infty} (H(x)-h)\,dx \quad . \tag{4.16}$$

From (4.14), we have

$$\frac{\partial}{\partial y}(p + g(y-h)) = -uu_y - vv_y = -uv_x - vv_y = -\operatorname{div}(v(u,v)) \quad ,$$

so that

$$\frac{\partial}{\partial y}(yp + gy(y-h)) = v^2 + p + g(y-h) - \operatorname{div}(yv(u,v)) \quad .$$

Hence,

$$\int_{-M}^{M} dx \int_0^{H(x)} (v^2 + p + g(y-h))\,dy = \int_0^{H(M)} yv(M,y)u(M,y)\,dy - \int_0^{H(-M)} yv(-M,y)u(-M,y)\,dy$$

$$+ \int_{-M}^{M} H(x)\{p(x,H(x)) + g(H(x)-h)\}dx$$

$$\to \int_{-\infty}^{\infty} H(x)\{p(x,H(x)) + g(H(x)-h)\}dx \quad \text{as} \quad M \to \infty$$

by (1.3). Since the pressure p is zero on $y = H(x)$, we have

$$\int_S (v^2 + p + g(y-h))\,dxdy = g\int_{-\infty}^{\infty} H(x)(H(x)-h)\,dx \quad . \tag{4.17}$$

The use of (4.16) and (4.17) with (4.14) then gives (4.12), where we have used the definition $F^2 = \frac{c^2}{gh}$.   q.e.d.

In Theorem 3.7, we showed that $F^2 \geq 1$ for any $(\mu,\theta) \in S$. Note that for the trivial solution $(\frac{6}{\pi},0) \in S$, the corresponding $F^2 = 1$. The following lemma shows that condition (b) of Theorem 4.8 is nearly satisfied; later we shall improve this result in Theorem 4.12 and prove that

$$\int_0^{\pi} (\pi-t)^{-1}(\ell n \frac{\pi}{\pi-t})^{\alpha} (\int_t^{\pi} (\pi-s)^{-1}\theta(s)ds)dt < \infty \text{ for all } \alpha \in [0,\tfrac{1}{2}) \quad ,$$

whence $F^2 > 1$ by Theorem 4.8.

LEMMA 4.9. Let $(\mu,\theta) \in S$ be non-trivial. Then there exists a constant $C > 0$ such that

$$\int_t^{\pi} (\pi-s)^{-1}\theta(s)ds \le C(\ell n \frac{\pi}{\pi-t})^{-1} + (\pi-t)^{-1} \int_t^{\pi} \theta(s)ds, \quad t \in (\tfrac{\pi}{2},\pi) \quad ,$$

and

$$\int_0^{\pi} (\pi-t)^{-2}(\int_t^{\pi} \theta(s)ds)dt < \infty \quad .$$

Proof. If $F^2 > 1$, then the result follows immediately from Theorem 4.7, and so we shall assume throughout this proof that $F^2 = 1$. The use of Theorem 2.5(f) yields

$$\theta(s) = \frac{1}{3\pi} \int_0^{\pi} (\ell n \left|\frac{2\pi-s-t}{s-t}\right|) \frac{f(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f \sin\theta} dt + H_1(s) \quad , \tag{4.18}$$

where $|H_1(s)| \le \text{const.}(\pi-s)$. Let $\beta \in (0,\pi)$ be given. Then

$$\int_{\beta}^{\pi} (\pi-s)^{-1}\theta(s)ds = \int_0^{\pi} Y(t;\beta) \frac{f(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f \sin\theta} dt + H_2(\beta) \quad , \tag{4.19}$$

where $|H_2(\beta)| \le \text{const.}(\pi-\beta)$ and

$$Y(t;\beta) = \frac{1}{3\pi} \int_{\beta}^{\pi}(\pi-s)^{-1}\ell n\left|\frac{2\pi-s-t}{s-t}\right|ds = \frac{1}{3\pi}\int_0^{(\pi-\beta)/(\pi-t)} x^{-1}\ell n\left|\frac{1+x}{1-x}\right|dx$$

$$= \frac{\pi}{6} - \frac{1}{3\pi}\int_{(\pi-\beta)/(\pi-t)}^{\infty} x^{-1}\ell n\left|\frac{1+x}{1-x}\right|dx \quad .$$

If we set

$$H_3(u) = \int_u^\infty x^{-1}\ell n\left|\frac{1+x}{1-x}\right|dx, \quad u \in (1,\infty) \quad ,$$

and

$$H_4(u) = \int_0^u x^{-1}\ell n\left|\frac{1+x}{1-x}\right|dx, \quad u \in (0,1) \quad ,$$

then (4.19) may be written as

$$\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds = \frac{\pi}{6}\int_\beta^\pi \frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f\sin\theta}dt - \frac{1}{3\pi}\int_\beta^\pi H_3\left(\frac{\pi-\beta}{\pi-t}\right)\frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f\sin\theta}dt$$

$$+ \frac{1}{3\pi}\int_0^\beta H_4\left(\frac{\pi-\beta}{\pi-t}\right)\frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f\sin\theta}dt + O(\pi-\beta) \quad . \tag{4.20}$$

Now

$$\frac{\pi}{6}\int_\beta^\pi \frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f\sin\theta}dt = -\frac{\pi}{6}\ell n(1-\frac{6}{\pi}\int_\beta^\pi f(t)\sin\theta(t)dt)$$

$$\geq \frac{\pi}{6}\{\frac{6}{\pi}\int_\beta^\pi f(t)\sin\theta(t)dt + \frac{18}{\pi^2}(\int_\beta^\pi f(t)\sin\theta(t)dt)^2\} = \int_\beta^\pi f(t)\sin\theta(t)dt + \frac{3}{\pi}(\int_\beta^\pi f(t)\sin\theta(t)dt)^2 \quad ,$$

where we have used the fact that $F^2 = 1$ and $-\ell n(1-x) \geq x + x^2/2$ for all $x \in (0,1)$. Use of this relation in (4.20) yields

$$\frac{3}{\pi}(\int_\beta^\pi f\sin\theta)^2 \leq \int_\beta^\pi(\pi-s)^{-1}\theta(s)ds - \int_\beta^\pi f(s)\sin\theta(s)ds + \frac{1}{3\pi}\int_\beta^\pi H_3\left(\frac{\pi-\beta}{\pi-t}\right)\frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f\sin\theta}dt$$

$$- \frac{1}{3\pi}\int_0^\beta H_4\left(\frac{\pi-\beta}{\pi-t}\right)\frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f\sin\theta}dt + O(\pi-\beta) \quad . \tag{4.21}$$

Now

$$\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds - \int_\beta^\pi f(s)\sin\theta(s)ds = \int_\beta^\pi (\pi-s)^{-1}(\theta(s) - \sin\theta(s))ds$$

$$+ \int_\beta^\pi \sin\theta(s)\{(\pi-s)^{-1} - f(s)\}ds \le \text{const.} \int_\beta^\pi (\pi-s)^{-1}\theta^3(s)ds + O(\pi-\beta) \quad ,$$

and the use of this in (4.21) gives

$$\frac{3}{\pi}\left(\int_\beta^\pi f\sin\theta\right)^2 \le \text{const.} \int_\beta^\pi (\pi-s)^{-1}\theta^3(s)ds + \frac{1}{3\pi}\int_\beta^\pi H_3\left(\frac{\pi-\beta}{\pi-t}\right) \frac{f(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f\sin\theta} dt$$

$$- \frac{1}{3\pi}\int_0^\beta H_4\left(\frac{\pi-\beta}{\pi-t}\right) \frac{f(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f\sin\theta} dt + O(\pi-\beta) \quad . \tag{4.22}$$

We now estimate the first term on the right of (4.22). From (4.18) there results

$$\theta(s) \le \text{const.}\left\{\int_0^\beta (\pi-t)^{-1}\ell n\left|\frac{2\pi-s-t}{s-t}\right|\theta(t)dt + \int_\beta^\pi (\pi-t)^{-1}\ell n\left|\frac{2\pi-s-t}{s-t}\right|\theta(t)dt + (\pi-s)\right\} \quad .$$

The Schwarz inequality gives

$$\int_\beta^\pi (\pi-t)^{-1}\ell n\left|\frac{2\pi-s-t}{s-t}\right|\theta(t)dt \le \left(\int_\beta^\pi (\pi-t)^{-1}\theta^2(t)dt\right)^{1/2} \left(\int_\beta^\pi (\pi-t)^{-1}\left(\ell n\left|\frac{2\pi-s-t}{s-t}\right|\right)^2 dt\right)^{1/2}$$

$$\le \left(\int_\beta^\pi (\pi-t)^{-1}\theta^2(t)dt\right)^{1/2} \left(\int_0^\infty x^{-1}\left(\ell n\left|\frac{1+x}{1-x}\right|\right)^2 dx\right)^{1/2} \le \text{const.}\left(\int_\beta^\pi (\pi-t)^{-1}\theta^2(t)dt\right)^{1/2} \quad , s \in (0,\pi) \quad ,$$

and

$$\int_0^\beta (\pi-t)^{-1}\ell n\left|\frac{2\pi-s-t}{s-t}\right|\theta(t)dt \le \left(\int_0^\beta (\pi-t)^{-2}\theta^2(t)dt\right)^{1/2} \left(\int_0^\beta \left(\ell n\left|\frac{2\pi-s-t}{s-t}\right|\right)^2 dt\right)^{1/2}$$

$$\le \left(\int_0^\beta (\pi-t)^{-2}\theta^2(t)dt\right)^{1/2} \left((\pi-s)\int_{(\pi-\beta)/(\pi-s)}^{\pi/(\pi-s)} \left(\ell n\left|\frac{1+x}{1-x}\right|\right)^2 dx\right)^{1/2}$$

$$\leq \text{const.}\ (\pi-\beta)^{1/2}\left(\int_0^\beta (\pi-t)^{-2}\theta^2(t)dt\right)^{1/2}, \quad s \in (\beta,\pi) \quad .$$

The use of these two inequalities and the boundedness of θ yields

$$\int_\beta^\pi (\pi-s)^{-1}\theta^3(s)ds \leq \text{const.} \int_\beta^\pi (\pi-s)^{-1}\theta(s)\{\int_\beta^\pi (\pi-t)^{-1}\theta^2(t)dt + (\pi-\beta)\int_0^\beta (\pi-t)^{-2}\theta(t)dt$$

$$+ (\pi-s)^2\}ds$$

$$\leq \text{const.}\{\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds \int_\beta^\pi (\pi-t)^{-1}\theta^2(t)dt + (\pi-\beta)\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds \int_0^\beta (\pi-t)^{-2}\theta(t)dt$$

$$+ (\pi-\beta)^2\} \quad . \tag{4.23}$$

If we use (4.23) and the fact that $f(t)\sin\theta(t) \geq \frac{2}{\pi}(\pi-t)^{-1}\theta(t)$, then (4.22) gives

$$\frac{12}{\pi^3}\left(\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds\right)^2$$

$$\leq \text{const.}\{\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds \int_\beta^\pi (\pi-t)^{-1}\theta^2(t)dt + (\pi-\beta)\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds \int_0^\beta (\pi-t)^{-2}\theta(t)dt\}$$

$$+ \frac{1}{3\pi}\int_\beta^\pi H_3\left(\frac{\pi-\beta}{\pi-t}\right)\frac{\frac{f(t)\sin\theta(t)}{t}}{\frac{1}{\mu}+\int_0^t f\sin\theta}dt - \frac{1}{3\pi}\int_0^\beta H_4\left(\frac{\pi-\beta}{\pi-t}\right)\frac{\frac{f(t)\sin\theta(t)}{t}}{\frac{1}{\mu}+\int_0^t f\sin\theta}dt + O(\pi-\beta) \quad , \tag{4.24}$$

and the constant is independent of β. Since $\theta(\pi) = 0$, for any value of the constant we may, and shall, restrict attention to those values of β sufficiently near to π for which

$$\text{const.}\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds \int_\beta^\pi (\pi-t)^{-1}\theta^2(t)dt \leq \frac{6}{\pi^3}\left(\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds\right)^2 \quad . \tag{4.25}$$

For $u \in (0,1)$, the function H_4 is given by

$$H_4(u) = \int_0^u x^{-1}\ln\left|\frac{1+x}{1-x}\right|dx$$

with

$$\frac{d}{du} H_4(u) = u^{-1} \ell n \left| \frac{1+u}{1-u} \right| \geq 2 \quad \text{for} \quad u \in (0,1) \quad .$$

Hence, $H_4(u) \geq 2u$, and so

$$\frac{1}{3\pi} \int_0^\beta H_4(\frac{\pi-\beta}{\pi-t}) \frac{f(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f \sin\theta} dt$$

$$\geq \frac{2}{3\pi}(\pi-\beta) \int_0^\beta \frac{f(t)\sin\theta(t)}{(\pi-t)(\frac{1}{\mu} + \int_0^t f\sin\theta)} dt \geq \frac{8}{\pi^3}(\pi-\beta) \int_0^\beta (\pi-t)^{-2}\theta(t)dt \quad .$$

Hence, for any value of the constant, and for any β in a sufficiently small neighborhood of π,

$$\text{const.}(\pi-\beta) \int_\beta^\pi (\pi-s)^{-1}\theta(s)ds \int_0^\beta (\pi-t)^{-2}\theta(t)dt - \frac{1}{3\pi}\int_0^\beta H_4(\frac{\pi-\beta}{\pi-t}) \frac{f(t)\sin\theta(t)}{\frac{1}{\mu} + \int_0^t f\sin\theta} dt \leq 0 \quad . \tag{4.26}$$

The use of (4.25) and (4.26) in (4.24) yields

$$(\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds)^2 \leq \text{const.}\int_\beta^\pi H_3(\frac{\pi-\beta}{\pi-t})(\pi-t)^{-1}\theta(t)dt + O(\pi-\beta) \quad ,$$

for all β sufficiently near to π, say, $\beta \in (\tau,\pi)$. Since $H_3(\frac{\pi-\beta}{\pi-t}) \leq \text{const.}(\frac{\pi-t}{\pi-\beta})$ for $t \in (\beta,\pi)$, it follows that for all $\beta \in (\tau,\pi)$,

$$(\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds)^2 \leq \text{const.}(\pi-\beta)^{-1} \int_\beta^\pi \theta(t)dt + \text{const.}(\pi-\beta) \quad .$$

Theorem 4.5 ensures that $(\pi-\beta)^{-1} \int_\beta^\pi \theta(t)dt \geq \text{const.}(\pi-\beta)\ell n(\frac{\pi}{\pi-\beta})$, and so

$$(\int_\beta^\pi (\pi-t)^{-1}\theta(t)dt)^2 \leq D(\pi-\beta)^{-1} \int_\beta^\pi \theta(t)dt \quad \text{for all} \quad \beta \in (\tau,\pi) \quad , \tag{4.27}$$

where D is a constant independent of β.

Define $g(s) = \int_s^\pi (\pi-t)^{-1}\theta(t)dt$, so that $\theta(s) = -(\pi-s)g'(s)$ and

$$(\pi-\beta)^{-1}\int_\beta^\pi \theta(t)dt = g(\beta) - (\pi-\beta)^{-1}\int_\beta^\pi g(s)ds \quad .$$

The use of this definition in (4.27) yields

$$(g(\beta))^2 \leq D\{g(\beta) - (\pi-\beta)^{-1}\int_\beta^\pi g(s)ds\} \quad ,$$

or, equivalently,

$$\int_\beta^\pi h(s)ds \leq (\pi-\beta)\{h(\beta) - (h(\beta))^2\} \quad ,$$

where $h = D^{-1}g$. If we set $f(\beta) = \int_\beta^\pi h(s)ds$, then

$$f(\beta) \leq (\pi-\beta)\{-f'(\beta) - (f'(\beta))^2\} \leq (\pi-\beta)(-f'(\beta)) \quad ,$$

or equivalently,

$$\frac{d}{d\beta}\left(\frac{f(\beta)}{\pi-\beta}\right) \leq -(\pi-\beta)^{-1}(f'(\beta))^2 \leq -(\pi-\beta)^{-1}\left(\frac{f(\beta)}{\pi-\beta}\right)^2 \quad .$$

Integrating this differential inequality yields

$$f(\beta) \leq (\pi-\beta)\left\{\ell n\left|\frac{\pi-\tau}{\pi-\beta}\right| + \frac{\pi-\tau}{f(\tau)}\right\}^{-1} \leq \text{const.}(\pi-\beta)\left(\ell n\left|\frac{\pi}{\pi-\beta}\right|\right)^{-1}, \quad \beta \in (\tau,\pi) \quad .$$

It follows from the definitions of f, g, and h that

$$\int_\beta^\pi \left(\int_s^\pi (\pi-t)^{-1}\theta(t)dt\right)ds \leq \text{const.}(\pi-\beta)\left(\ell n\left|\frac{\pi}{\pi-\beta}\right|\right)^{-1}, \quad \beta \in (\tau,\pi) \quad ,$$

and an integration by parts then gives

$$\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds \leq \text{const.}\left(\ell n\left|\frac{\pi}{\pi-\beta}\right|\right)^{-1} + (\pi-\beta)^{-1}\int_\beta^\pi \theta(s)ds \quad .$$

To complete the proof of the theorem, it suffices to show that if

$A(t) = (\pi-t)^{-1} \int_t^{\pi} \theta(s)ds$, then

$$\int_0^{\pi} (\pi-t)^{-1} A(t)dt < \infty .$$

An integration by parts gives

$$\int_0^{\pi-\varepsilon} (\pi-t)^{-1}\theta(t)dt = -\varepsilon^{-1} \int_{\pi-\varepsilon}^{\pi} \theta(t)dt + \pi^{-1} \int_0^{\pi} \theta(t)dt + \int_0^{\pi-\varepsilon} (\pi-t)^{-2}\left(\int_t^{\pi} \theta(s)ds\right)dt ,$$

and the result follows immediately upon letting $\varepsilon \to 0$. q.e.d.

The following results will be needed in the proof of Lemma 4.11.

LEMMA 4.10. Let (μ,θ) be a non-trivial element of S. Then

(a) $\int_0^{\pi} (\pi-t)^{-2}\left(\int_t^{\pi} \theta(s)ds\right)dt < \infty$,

(b) $(\pi-t)^{-1} \int_t^{\pi} \theta(s)ds \to 0$ as $t \to \pi$,

(c) $\liminf_{t\to\pi} (\pi-t)^{-1} \ln\left(\frac{\pi}{\pi-t}\right) \int_t^{\pi} \theta(s)ds = 0$,

(d) $\int_0^{\pi}\left(\int_0^{t} (\pi-s)^{-2}\theta(s)ds\right)dt < \infty$,

(e) $(\pi-t) \int_0^{t} (\pi-s)^{-2}\theta(s)ds \to 0$ as $t \to \pi$,

(f) $\liminf_{t\to\pi} (\pi-t) \ln\left(\frac{\pi}{\pi-t}\right) \int_0^{t} (\pi-s)^{-2}\theta(s)ds = 0$.

Proof. The results follow easily from the fact that $\int_0^{\pi} (\pi-t)^{-1}\theta(t)dt < \infty$ by integrating by parts (cf. the arguments at the end of the proof of Lemma 4.9).

q.e.d.

LEMMA 4.11. <u>Let</u> $(\mu,\theta) \in S$ <u>be non-trivial</u>. <u>Then for each</u> $\gamma \in (0,1)$,

(a) $\int_0^\pi (\pi-t)^{-1}(\ln \frac{\pi}{\pi-t})^\gamma \theta(t)dt < \infty$,

(b) $\int_0^\pi (\pi-t)^{-2}(\ln \frac{\pi}{\pi-t})^\gamma (\int_t^\pi \theta(s)ds)dt < \infty$,

(c) $(\pi-t)^{-1}(\ln \frac{\pi}{\pi-t})^\gamma \int_t^\pi \theta(s)ds \to 0$ <u>as</u> $t \to \pi$,

(d) $\liminf_{t\to\pi} (\pi-t)^{-1}(\ln \frac{\pi}{\pi-t})^{1+\gamma}\int_t^\pi \theta(s)ds = 0$,

(e) $\int_0^\pi (\ln \frac{\pi}{\pi-t})^\gamma (\int_0^t (\pi-s)^{-2}\theta(s)ds)dt < \infty$,

(f) $(\pi-t)(\ln \frac{\pi}{\pi-t})^\gamma \int_0^t (\pi-s)^{-2}\theta(s)ds \to 0$ <u>as</u> $t \to \pi$,

(g) $\liminf_{t\to\pi} (\pi-t)(\ln \frac{\pi}{\pi-t})^{1+\gamma}\int_0^t (\pi-s)^{-2}\theta(s)ds = 0$.

<u>Proof</u>. Lemmas 4.9 and 4.10(a) show that

$$\int_0^\pi (\pi-t)^{-1}(\ln \frac{\pi}{\pi-t})^{-\gamma}(\int_t^\pi (\pi-s)^{-1}\theta(s)ds)dt < \infty \text{ for all } \gamma \in (0,1) .$$

For $\delta \in (0,\pi)$, an integration by parts gives

$$\int_0^{\pi-\delta} (\pi-t)^{-1}(\ln \frac{\pi}{\pi-t})^{-\gamma}(\int_t^\pi (\pi-s)^{-1}\theta(s)ds)dt = (1-\gamma)^{-1}(\ln \frac{\pi}{\delta})^{1-\gamma}\int_{\pi-\delta}^\pi (\pi-s)^{-1}\theta(s)ds$$

$$+ (1-\gamma)^{-1}\int_0^{\pi-\delta} (\pi-t)^{-1}(\ln \frac{\pi}{\pi-t})^{1-\gamma}\theta(t)dt . \tag{4.28}$$

By Lemmas 4.9 and 4.10(c), we have

$$\liminf_{\delta\to 0} (\ln \tfrac{\pi}{\delta})^{1-\gamma} \int_{\pi-\delta}^{\pi} (\pi-s)^{-1}\theta(s)\,ds \le \liminf_{\delta\to 0} \text{const.}(\ln \tfrac{\pi}{\delta})^{-\gamma}$$

$$+ \liminf_{\delta\to 0} (\ln \tfrac{\pi}{\delta})^{1-\gamma}\delta^{-1}\int_{\pi-\delta}^{\pi} \theta(s)\,ds = 0 \quad ,$$

and the use of this in (4.28) then gives (a).

(b) - (d) Now

$$\int_0^{\pi-\delta} (\pi-t)^{-1}(\ln \tfrac{\pi}{\pi-t})^{\gamma}\theta(t)\,dt = -\delta^{-1}(\ln \tfrac{\pi}{\delta})^{\gamma}\int_{\pi-\delta}^{\pi} \theta(s)\,ds + \int_0^{\pi-\delta} (\pi-t)^{-2}(\ln \tfrac{\pi}{\pi-t})^{\gamma}(\int_t^{\pi} \theta(s)\,ds)\,dt$$

$$+ \gamma \int_0^{\pi-\delta} (\pi-t)^{-2}(\ln \tfrac{\pi}{\pi-t})^{\gamma-1}(\int_t^{\pi} \theta(s)\,ds)\,dt \quad , \tag{4.29}$$

and (b) follows from the use of Lemma 4.10(c) in (4.29). Equation (4.29) then implies that

$$(\pi-t)^{-1}(\ln \tfrac{\pi}{\pi-t})^{\gamma}\int_t^{\pi} \theta(s)\,ds$$

has a limit as $t \to \pi$, and Lemma 4.10(c) ensures that this limit is zero. Part (d) follows from the fact that

$$\int_0^{\pi} (\pi-t)^{-1}(\ln \tfrac{\pi}{\pi-t})^{-1}\{(\pi-t)^{-1}(\ln \tfrac{\pi}{\pi-t})^{1+\gamma}\int_t^{\pi} \theta(s)\,ds\}\,dt < \infty \quad .$$

(e) - (g) An integration by parts gives

$$\int_0^{\pi-\delta} (\pi-t)(\ln \tfrac{\pi}{\pi-t})^{\gamma}\{(\pi-t)^{-2}\theta(t)\}\,dt = \delta(\ln \tfrac{\pi}{\delta})^{\gamma}\int_0^{\pi-\delta} (\pi-s)^{-2}\theta(s)\,ds$$

$$+ \int_0^{\pi-\delta} (\ln \tfrac{\pi}{\pi-t})^{\gamma}(\int_0^{t} (\pi-s)^{-2}\theta(s)\,ds)\,dt - \gamma \int_0^{\pi-\delta} (\ln \tfrac{\pi}{\pi-t})^{\gamma-1}(\int_0^{t} (\pi-s)^{-2}\theta(s)\,ds)\,dt \quad . \tag{4.30}$$

The use of Lemma 4.10(f) immediately gives (e), and (f) and (g) then follow from (4.30). q.e.d.

THEOREM 4.12. Let $(\mu,\theta) \in S$ be non-trivial. Then for all $\alpha \in [0,1/2)$,

$$\text{(a)} \quad \int_0^\pi (\pi-t)^{-1}\left(\ln \frac{\pi}{\pi-t}\right)^\alpha \left(\int_t^\pi (\pi-s)^{-1}\theta(s)ds\right)dt < \infty .$$

$$\text{(b)} \quad F^2 = \frac{6}{\pi}\left(\frac{1}{\mu} + \int_0^\pi f(t)\sin\theta(t)dt\right) > 1 .$$

Proof. (a) Theorem 4.7 ensures that we need only consider the case $F^2 = 1$. Since $f(t)\sin\theta(t) \geq \frac{2}{\pi}(\pi-t)^{-1}\theta(t)$, equation (4.22) gives

$$\frac{12}{\pi^3}\left(\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds\right)^2 \leq \text{const.}\int_\beta^\pi (\pi-s)^{-1}\theta^3(s)ds + \frac{1}{3\pi}\int_\beta^\pi H_3\left(\frac{\pi-\beta}{\pi-t}\right)N\theta(t)dt$$

$$- \frac{1}{3\pi}\int_0^\beta H_4\left(\frac{\pi-\beta}{\pi-t}\right)N\theta(t)dt + O(\pi-\beta), \quad \beta \in (0,\pi) \quad , \tag{4.31}$$

where $N\theta(t) = f(t)\sin\theta(t)\left(\frac{1}{\mu} + \int_0^t f(w)\sin\theta(w)dw\right)^{-1}$. Since $\theta(\pi) = 0$, equation (4.23) shows that

$$\text{const.}\int_\beta^\pi (\pi-s)^{-1}\theta^3(s)ds$$

$$\leq \frac{6}{\pi^3}\left(\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds\right)^2 + \text{const.}(\pi-\beta)\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds \int_0^\beta (\pi-t)^{-2}\theta(t)dt$$

$$+ \text{const.}(\pi-\beta)^2$$

for all β sufficiently near to π, for $\beta \in (\tau,\pi)$, say. The use of this in (4.31) yields

$$\frac{6}{\pi^3}\left(\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds\right)^2 \leq \text{const.}A(\beta) + \frac{1}{3\pi}\int_\beta^\pi H_3\left(\frac{\pi-\beta}{\pi-t}\right)N\theta(t)dt - \frac{1}{3\pi}\int_0^\beta H_4\left(\frac{\pi-\beta}{\pi-t}\right)N\theta(t)dt$$

$$+ \text{const.}(\pi-\beta) \tag{4.32}$$

for all $\beta \in (\tau,\pi)$, where

$$A(\beta) = (\pi-\beta)\int_\beta^\pi (\pi-s)^{-1}\theta(s)ds \int_0^\beta (\pi-t)^{-2}\theta(t)dt \quad .$$

Lemmas 4.9, 4.11(c), and 4.11(e) ensure that

$$\int_0^{\pi} (\pi-s)^{-1}\left(\ln \frac{\pi}{\pi-s}\right)^{1+\gamma} |A(s)|\,ds < \infty \quad \text{for all} \quad \gamma \in (0,1) \quad . \tag{4.33}$$

For each $\beta \in (0,\pi)$, set

$$\Gamma(\beta) = \frac{1}{2}\int_{\beta}^{\pi} H_3\left(\frac{\pi-\beta}{\pi-t}\right) N\theta(t)\,dt - \frac{1}{2}\int_0^{\beta} H_4\left(\frac{\pi-\beta}{\pi-t}\right) N\theta(t)\,dt \quad ,$$

so that (4.32) may be rewritten as

$$\frac{6}{\pi^3}\left(\int_{\beta}^{\pi} (\pi-s)^{-1}\theta(s)\,ds\right)^2 \le \text{const.}A(\beta) + \frac{2}{3\pi}\Gamma(\beta) + \text{const.}(\pi-\beta) \tag{4.34}$$

for all $\beta \in (\tau,\pi)$. We now show that

$$\liminf_{\delta\to 0} \int_0^{\pi-\delta} (\pi-s)^{-1}\left(\ln \frac{\pi}{\pi-s}\right)^{1+\gamma}\Gamma(s)\,ds < \infty \quad \text{for all} \quad \gamma \in (0,1) \quad .$$

A simple calculation yields

$$H_4(u) \equiv \int_0^u x^{-1}\ln\left|\frac{1+x}{1-x}\right| dx = 2\sum_{k=0}^{\infty}(2k+1)^{-2}u^{2k+1}, \quad u \in [0,1] \quad ,$$

and $H_3(u) = H_4(u^{-1})$, $u \in [1,\infty)$, whence

$$\Gamma(\beta) = \sum_{k=0}^{\infty}(2k+1)^{-2}\left\{(\pi-\beta)^{-2k-1}\int_{\beta}^{\pi}(\pi-t)^{2k+1}N\theta(t)\,dt - (\pi-\beta)^{2k+1}\int_0^{\beta}(\pi-t)^{-2k-1}N\theta(t)\,dt\right\} \quad .$$

It follows that, for each $\delta \in (0,\pi)$,

$$\int_0^{\pi-\delta} (\pi-\beta)^{-1}\left(\ln\frac{\pi}{\pi-\beta}\right)^{1+\gamma}\Gamma(\beta)\,d\beta$$

$$= \sum_{k=0}^{\infty}(2k+1)^{-2}\left\{\int_0^{\pi-\delta}(\pi-\beta)^{-2k-2}\left(\ln\frac{\pi}{\pi-\beta}\right)^{1+\gamma}\left(\int_{\beta}^{\pi}(\pi-t)^{2k+1}N\theta(t)\,dt\right)d\beta\right.$$

$$\left. - \int_0^{\pi-\delta}(\pi-\beta)^{2k}\left(\ln\frac{\pi}{\pi-\beta}\right)^{1+\gamma}\left(\int_0^{\beta}(\pi-t)^{-2k-1}N\theta(t)\,dt\right)d\beta\right\} \quad . \tag{4.35}$$

Integrations by parts give

$$\int_0^{\pi-\delta} (\pi-\beta)^{-2k-2}(\ln \frac{\pi}{\pi-\beta})^{1+\gamma}(\int_\beta^\pi (\pi-t)^{2k+1}N\theta(t)dt)d\beta = (2k+1)^{-1} \int_0^{\pi-\delta} (\ln \frac{\pi}{\pi-\beta})^{1+\gamma}N\theta(\beta)d\beta$$

$$+ \int_0^{\pi-\delta} (\pi-\beta)^{2k+1}f_k(\beta)N\theta(\beta)d\beta + (2k+1)^{-1}\delta^{-2k-1}(\ln \frac{\pi}{\delta})^{1+\gamma} \int_{\pi-\delta}^{\pi} (\pi-t)^{2k+1}N\theta(t)dt$$

$$+ f_k(\pi-\delta)\int_{\pi-\delta}^{\pi} (\pi-t)^{2k+1}N\theta(t)dt \quad , \tag{4.36}$$

where

$$f_k(\beta) = -(1+\gamma)(2k+1)^{-1} \int_0^{\beta}(\pi-t)^{-2k-2}(\ln \frac{\pi}{\pi-t})^{\gamma}dt$$

and

$$|f_k(\beta)| \le \text{const.}(2k+1)^{-1}(\pi-\beta)^{-2k-1}(\ln \frac{\pi}{\pi-\beta})^{\gamma}, \quad \beta \in (0,\pi) \quad , \tag{4.37}$$

where the constant is independent of k and δ. Lemma 4.11(c) gives

$$|f_k(\pi-\delta)\int_{\pi-\delta}^{\pi} (\pi-t)^{2k+1}N\theta(t)dt| \le \text{const.}(2k+1)^{-1}\delta^{-1}(\ln \frac{\pi}{\delta})^{\gamma} \int_{\pi-\delta}^{\pi} (\frac{\pi-t}{\delta})^{2k}\sin \theta(t)dt$$

$$\le \text{const.}(2k+1)^{-1} \tag{4.38}$$

and the constant is independent of k and δ. Now for the other terms in (4.35); integrations by parts give

$$-\int_0^{\pi-\delta} (\pi-\beta)^{2k}(\ln \frac{\pi}{\pi-\beta})^{1+\gamma}(\int_0^{\beta} (\pi-t)^{-2k-1}N\theta(t)dt)d\beta = -(2k+1)^{-1}\int_0^{\pi-\delta} (\ln \frac{\pi}{\pi-\beta})^{1+\gamma}N\theta(\beta)d\beta$$

$$- \int_0^{\pi-\delta} (\pi-\beta)^{-2k-1}g_k(\beta)N\theta(\beta)d\beta + (2k+1)^{-1}\delta^{2k+1}(\ln \frac{\pi}{\delta})^{1+\gamma} \int_0^{\pi-\delta} (\pi-t)^{-2k-1}N\theta(t)dt$$

$$+ g_k(\pi-\delta)\int_0^{\pi-\delta} (\pi-t)^{-2k-1}N\theta(t)dt \quad , \tag{4.39}$$

where

$$g_k(\beta) = (1+\gamma)(2k+1)^{-1} \int_\beta^{\pi}(\pi-t)^{2k}(\ln \frac{\pi}{\pi-t})^{\gamma}dt \quad ,$$

and there exists a constant (independent of k and δ) such that

$$|g_k(\beta)| \le \text{const.}(2k+1)^{-1}(\pi-\beta)^{2k+1}\ln(\frac{\pi}{\pi-\beta})^{\gamma}, \quad \beta \in (0,\pi) \quad . \tag{4.40}$$

Lemma 4.11(f) gives

$$|g_k(\pi-\delta)\int_0^{\pi-\delta} (\pi-t)^{-2k-1}N\theta(t)dt| \le \text{const.}(2k+1)^{-1}\delta(\ln \frac{\pi}{\delta})^{\gamma}\int_0^{\pi-\delta} (\frac{\delta}{\pi-t})^{2k}(\pi-t)^{-2}\sin \theta(t)dt$$

$$\le \text{const.}(2k+1)^{-1} \quad , \tag{4.41}$$

and the constant is independent of k and δ. The use of (4.36), (4.38), (4.39), and (4.41) in (4.35) gives

$$\int_0^{\pi-\delta}(\pi-\beta)^{-1}\left(\ln\frac{\pi}{\pi-\beta}\right)^{1+\gamma}\Gamma(\beta)\,d\beta = \sum_{k=0}^{\infty}(2k+1)^{-2}\Big\{\int_0^{\pi-\delta}(\pi-\beta)^{2k+1}f_k(\beta)N\theta(\beta)\,d\beta$$

$$+ (2k+1)^{-1}\delta^{-2k-1}\left(\ln\frac{\pi}{\delta}\right)^{1+\gamma}\int_{\pi-\delta}^{\pi}(\pi-t)^{2k+1}N\theta(t)\,dt - \int_0^{\pi-\delta}(\pi-\beta)^{-2k-1}g_k(\beta)N\theta(\beta)\,d\beta$$

$$+ (2k+1)^{-1}\delta^{2k+1}\left(\ln\frac{\pi}{\delta}\right)^{1+\gamma}\int_0^{\pi-\delta}(\pi-t)^{-2k-1}N\theta(t)\,dt + O\left(\frac{1}{2k+1}\right)\Big\} \quad . \tag{4.42}$$

Equation (4.37) ensures that

$$\left|\int_0^{\pi-\delta}(\pi-\beta)^{2k+1}f_k(\beta)N\theta(\beta)\,d\beta\right| \le \text{const.}(2k+1)^{-1}\int_0^{\pi-\delta}(\pi-\beta)^{-1}\left(\ln\frac{\pi}{\pi-\beta}\right)^{\gamma}\sin\theta(\beta)\,d\beta$$

$$\le \text{const.}(2k+1)^{-1}$$

by Lemma 4.11(a), and the constant is independent of k and δ. Similarly

$$\left|\int_0^{\pi-\delta}(\pi-\beta)^{-2k-1}g_k(\beta)\,N\theta(\beta)\,d\beta\right| \le \text{const.}(2k+1)^{-1} \quad .$$

The use of these two inequalities in (4.42) yields

$$\int_0^{\pi-\delta}(\pi-\beta)^{-1}\left(\ln\frac{\pi}{\pi-\beta}\right)^{1+\gamma}\Gamma(\beta)\,d\beta \le \text{const.} + \sum_{k=0}^{\infty}(2k+1)^{-3}\delta^{-2k-1}\left(\ln\frac{\pi}{\delta}\right)^{1+\gamma}\int_{\pi-\delta}^{\pi}(\pi-t)^{2k+1}N\theta(t)\,dt$$

$$+ \sum_{k=0}^{\infty}(2k+1)^{-3}\delta^{2k+1}\left(\ln\frac{\pi}{\delta}\right)^{1+\gamma}\int_0^{\pi-\delta}(\pi-t)^{-2k-1}N\theta(t)\,dt$$

$$\le \text{const.} + \text{const.}\Big\{\delta^{-1}\left(\ln\frac{\pi}{\delta}\right)^{1+\gamma}\int_{\pi-\delta}^{\pi}\theta(t)\,dt + \delta\left(\ln\frac{\pi}{\delta}\right)^{1+\gamma}\int_0^{\pi-\delta}(\pi-t)^{-2}\theta(t)\,dt\Big\} \quad . \tag{4.43}$$

Lemma 4.11 shows that

$$\int_0^{\pi}(\pi-t)^{-1}\left(\ln\frac{\pi}{\pi-t}\right)^{-1}\Big\{(\pi-t)^{-1}\left(\ln\frac{\pi}{\pi-t}\right)^{1+\gamma}\int_t^{\pi}\theta(s)\,ds + (\pi-t)\left(\ln\frac{\pi}{\pi-t}\right)^{1+\gamma}\int_0^{t}(\pi-s)^{-2}\theta(s)\,ds\Big\}dt < \infty ,$$

and so

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FINITE-AMPLITUDE SOLITARY WATER WAVES.(U)

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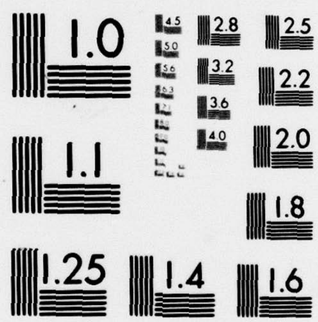

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$$\liminf_{\delta\to 0} \{\delta^{-1}(\ln \frac{\pi}{\delta})^{1+\gamma} \int_{\pi-\delta}^{\pi} \theta(s)ds + \delta(\ln \frac{\pi}{\delta})^{1+\gamma} \int_{0}^{\pi-\delta} (\pi-s)^{-2}\theta(s)ds\} = 0 \quad .$$

The use of this estimate in (4.43) gives

$$\liminf_{\delta\to 0} \int_{0}^{\pi-\delta} (\pi-\beta)^{-1}(\ln \frac{\pi}{\pi-\beta})^{1+\gamma}\Gamma(\beta)d\beta < \infty, \text{ for all } \gamma \in (0,1) \quad .$$

The use of this result with (4.33) in (4.34) proves that

$$\int_{0}^{\pi} (\pi-\beta)^{-1}(\ln \frac{\pi}{\pi-\beta})^{1+\gamma}(\int_{\beta}^{\pi} (\pi-s)^{-1}\theta(s)ds)^{2}d\beta < \infty, \text{ for all } \gamma \in (0,1) \quad .$$

Part (a) of this theorem follows easily by the Schwarz inequality.

(b) Part (b) follows from Theorem 4.8. q.e.d.

4.6. Solitary waves of depression

Assume that

$$\left.\begin{array}{l} \theta \text{ is an odd continuous function on } [-\pi,\pi] \text{ with } 0 < \theta(s) < \frac{\pi}{2} \\ \text{for } s \in (0,\pi),\ \theta(\pi) = 0, \text{ and } \frac{1}{\mu} - \int_{0}^{\pi} f(w)\sin\theta(w)dw > 0 \quad . \end{array}\right\} \quad (4.44)$$

Suppose moreover that θ satisfies the integral equation

$$\theta(s) = \frac{2}{3}\int_{0}^{\pi} G(s,t) \frac{\frac{f(t)\sin\theta(t)}{t}}{\frac{1}{\mu} - \int_{0} f \sin\theta} dt, \quad s \in (0,\pi) \quad . \quad (4.45)$$

One can easily show that any such θ must be analytic on $(-\pi,\pi)$. By following the arguments for the proof of Theorem 1.1, one can show that there exists a complex potential $\tilde{w}$ and a free surface Γ satisfying (1.3) - (1.8a) with (1.8b) replaced by the condition that H be strictly increasing for $x > 0$. Such a wave would be a non-trivial solitary wave of depression. We now show that there does not exist a function θ satisfying (4.44) and (4.45). If such a function does exist, then the arguments in the proof of Theorem 3.7(c) yield

$$\frac{1}{\mu} - \int_0^\pi f(w)\sin\theta(w)\,dw \geq \frac{\pi}{6} .$$

Since $\theta(0) = \theta(\pi) = 0$, there exists $\tilde{s} \in (0,\pi)$ such that $\theta(\tilde{s}) = |\theta|_{C[0,\pi]}$, and so

$$|\theta|_{C[0,\pi]} = \theta(\tilde{s}) = \frac{2}{3}\int_0^\pi G(\tilde{s},t)\,\frac{f(t)\sin\theta(t)}{\frac{1}{\mu} - \int_0^t f\sin\theta}\,dt < \frac{4}{\pi}\int_0^\pi G(\tilde{s},t)f(t)\theta(t)\,dt$$

$$< \frac{4}{\pi}|\theta|_{C[0,\pi]}\int_0^\pi G(\tilde{s},t)\left(\frac{1}{2}\tan\frac{t}{2} + \frac{1}{2}\cot\frac{t}{2}\right)dt = |\theta|_{C[0,\pi]} , \qquad (4.46)$$

where the final equality follows from Theorem 2.5(e). The strict inequality in (4.46) is a contradiction, and so there does not exist a function θ satisfying (4.44) and (4.45). This result agrees with the statement of Scott Russell reported in section 1.1 that 'a negative wave is soon broken up and dissipated'.

5. SOLITARY WAVES OF GREATEST HEIGHT

5.1. The existence of a solitary wave of greatest height

The notation here is the same as that of section 3. We now prove the following result.

THEOREM 5.1. *If* $\{(\mu_n, \theta_n)\} \subset \mathcal{C}$ *and* $\mu_n \to \infty$ *as* $n \to \infty$, *then there exists a subsequence* $\{\theta_{n_k}\}$ *of* $\{\theta_n\}$ *which converges in* $C[\delta,\pi]$ *for each* $\delta \in (0,\pi)$ *to a non-zero solution* θ *of the equation*

$$\theta(s) = \frac{2}{3}\int_0^{\pi} G(s,t)\,\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)\,dw}\,dt,\ s \in (0,\pi]\ , \tag{5.1}$$

and $0 < \theta(s) < \frac{\pi}{2}$ *on* $(0,\pi)$.

Remark. Theorem 5.1 concerns the existence of a solitary water wave in the case when $\frac{1}{\mu} = q_c = 0$. This is a solitary wave which has a stagnation point at its crest and is variously described as a wave of greatest height, or a wave of extreme form [25]. Although all the evidence suggests that if θ satisfies (5.1) and is positive on $(0,\pi)$, then $\lim_{s\to 0+}\theta(s) = \frac{\pi}{6}$, this proposition, which was first conjectured by Stokes [43], has so far eluded proof.

Proof of Theorem 5.1. The capital letters (A), (B), etc. below refer to various subsections in the proof of Theorem 3.8. Without loss of generality, identify θ_n with its odd extension to $(-\pi,\pi)$ and let $\theta_n \rightharpoonup \theta$, $\sin\theta_n \rightharpoonup \sigma$ and $\frac{1}{\mu_n} \to 0$ as $n \to \infty$. As in the proof of Theorem 3.7(c), one can show that there exists a $\delta > 0$ such that

$$\theta_n(s) \geq \delta \sin s \tag{5.2}$$

for $s \in (0,\pi)$ and for all positive integers n, whence $\theta(s) \geq \delta \sin s$ almost everywhere on $(0,\pi)$. If ρ_n is given in terms of (μ_n,θ_n) by (3.40), and ρ is defined by (3.43) with $\frac{1}{\mu} = 0$, then the use of (5.2) gives

$$-\frac{1}{3}\ln\left(\frac{\delta}{\pi}\int_0^t \sin s\, ds\right) \geq \rho_n(t) \geq -\frac{1}{3}\ln\left(\frac{1}{\mu_n} + \frac{\pi}{2}\ln\left(\frac{\pi}{\pi-t}\right)\right) \quad .$$

The method of (A) now implies that $\theta_n \to \theta$ and $\rho_n \to \rho$ in $L_2(-\pi,\pi)$, as $n \to \infty$, and that $\sigma = \sin\theta$. From (5.2) it is clear that $\{\rho_n\}$ is bounded in $C^{0,\alpha}[\delta,\pi-\delta]$ for each $\delta \in (0,\frac{\pi}{2})$, and so $\theta_n \to \theta$ in $C[\delta,\pi-\delta]$ for each $\delta \in (0,\frac{\pi}{2})$, exactly as in (B). By the arguments of (C) and (D), one can show that $\int_0^\pi f(t)\sin\theta(t)dt$ is finite, that θ satisfies (5.1) on $(0,\pi)$, and hence that $\lim_{s\to\pi}\theta(s) = 0$. (Since $\{\theta_n\}$ has <u>not</u> been shown to converge to θ uniformly on $[0,\pi-\delta)$ for $\delta \in (0,\pi)$, it does not follow that $\theta \in K_0$. In Theorem 5.2 we show that $\theta \notin K_0$.)

However, the facts that $\theta \in C(\delta,\pi]$ for each $\delta \in (0,\pi)$ and $\theta(\pi) = 0$ ensure, as in Theorem 3.7(c), that $\int_0^\pi f(t)\sin\theta(t)dt \geq \frac{\pi}{6}$. We may then conclude, by the methods of (F) and (G), that $\frac{1}{\mu_n} + \int_0^\pi f(t)\sin\theta_n(t)dt \to \int_0^\pi f(t)\sin\theta(t)dt$, and that $\theta_n \to \theta$ in $C[\delta,\pi]$ for each $\delta \in (0,\pi]$. Since $\theta_n(s) \leq \frac{\pi}{2}$ for each n, then the same holds for θ. By the proof of Theorem 3.7(b), $\theta(s) < \frac{\pi}{2}$ on $(0,\pi)$.

q.e.d.

5.2. <u>Properties of a solitary wave of greatest height and Stokes' conjecture</u>

We begin with a general result about the regularity of solutions of (5.1).

THEOREM 5.2. <u>If</u> $\theta \in C(0,\pi)$ <u>is a solution of</u> (5.1) <u>and</u> $0 < \theta(s) \leq \pi$ <u>on</u> $(0,\pi)$, <u>then</u> (a) $\theta \in C^{1,\alpha}[a,b]$ <u>for each interval</u> $[a,b] \subset (0,\pi)$ <u>and for each</u> $\alpha \in (0,1)$, <u>and consequently</u> θ <u>is analytic on</u> $(0,\pi)$.

(b) $0 < \theta(s) < \frac{\pi}{2}$ <u>for all</u> $s \in (0,\pi)$.

(c) $\frac{\pi}{6} \leq \int_0^\pi f(t)\sin\theta(t)dt \leq M$, <u>where</u> M <u>is a constant independent of</u> θ, $\theta \in C(0,\pi]$ <u>and</u> $\theta(\pi) = 0$.

(d) <u>There exists a constant</u> $\alpha > 0$ <u>such that</u> $\theta(s) \geq \frac{\alpha}{s}\int_0^s \sin\theta(t)dt$, <u>for all</u> $s \in (0,\pi/2)$.

(e) <u>There exists a constant</u> $\beta > 0$ <u>such that for all</u> $\varepsilon \in (0,\pi/2)$

$$\frac{1}{\varepsilon}\int_0^{\varepsilon}\theta(s)ds \geq \beta\varepsilon\int_{\varepsilon}^{\pi/2}\frac{\sin\theta(t)}{t\int_0^t \sin\theta(w)dw}dt \quad .$$

(f) $\liminf_{s\to 0+}\theta(s) > 0.$

(g) Either $\lim_{s\to 0+}\theta(s) = \frac{\pi}{6}$, or

$$\liminf_{s\to 0+}\frac{1}{s}\int_0^s \sin\theta(t)dt < \frac{1}{2} < \limsup_{s\to 0+}\frac{1}{s}\int_0^s \sin\theta(t)dt \quad .$$

Proof. (a) If $\delta \in (0,\frac{\pi}{4})$, then for all $s \in (0,\pi)$

$$\theta(s) = \frac{2}{3}\{\int_0^{\delta/2} G(s,t)\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw}dt + \int_{\pi-\frac{\delta}{2}}^{\pi} G(s,t)\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw}dt$$

$$+ \int_{\delta/2}^{\pi-\delta/2} G(s,t)\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw}dt\} \ .$$

It is clear from Theorem 2.5(h) that the result will follow when we prove that the function χ defined on $[\delta,\pi-\delta]$ by

$$\chi(s) = \int_0^{\delta/2} G(s,t)\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw}dt + \int_{\pi-\frac{\delta}{2}}^{\pi} G(s,t)\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw}dt$$

is in $C^{1,\alpha}[\delta,\pi-\delta]$. It is shown in the proof of Theorem 2.5(c) that for $(s,t) \in [0,\pi]\times[0,\pi]$ and $s \neq t$,

$$\frac{\partial^2 G}{\partial s^2} = \frac{1}{2\pi}\frac{\sin s\sin t}{(\cos s - \cos t)^2} \quad ,$$

whence for $s \in [\delta,\pi-\delta]$ and $t \in (0,\delta/2] \cup [\pi-\delta/2,\pi)$,

$$0 \leq \frac{\partial^2 G}{\partial s^2} \leq \text{const.}\ \sin s\sin t \quad ,$$

where the constant depends on δ. Therefore for $s \in [\delta, \pi-\delta]$,

$$0 \le \frac{d^2\chi}{ds^2} \le \text{const.}\left\{\int_0^{\delta/2} \frac{\sin t\, f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt + \int_{\pi-\frac{\delta}{2}}^{\pi} \frac{\sin t\, f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw}\right\}\sin s$$

$$\le \text{const.}\left\{\int_0^{\pi} \frac{\sin t\, f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt\right\}\sin s = \text{const.} \sin s \quad .$$

Therefore, if $[a,b] \subset (0,\pi)$ and $\alpha \in (0,1)$, $\theta \in C^{1,\alpha}[a,b]$. The result of (a) now follows from Lewy's theorem [28], exactly as in Theorem 4.3.

(b) With the regularity of θ established in (a), the method used in the proof of Theorem 3.7(b) can be used to prove (b).

(c) It will suffice to show that $\int_{\pi/2}^{\pi} (\pi-s)^{-1}\theta(s)ds$ is finite in order to show that $\int_0^{\pi} f(t)\sin\theta(t)dt$ is finite. Since θ satisfies (5.1), the use of Fubini's theorem yields that for $\delta \in (0,\pi/2)$,

$$\int_{\pi/2}^{\pi-\delta} \frac{\theta(s)}{(\pi-s)} ds = \int_0^{\pi} V(\delta;t) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt \quad , \tag{5.3}$$

where, by Theorem 2.5(e), for $(\delta,t) \in (0,\pi/2) \times [0,\pi]$

$$V(\delta;t) = \frac{2}{3}\int_{\pi/2}^{\pi-\delta} \frac{G(s,t)}{(\pi-s)} ds$$

$$\le \text{const. } t \quad , \tag{5.4}$$

the constant being independent of (δ,t). It follows from the use of Theorem 2.5(e) that

$$V(\delta;t) \le \frac{1}{3\pi}\int_{\pi/2}^{\pi-\delta} \left(\ell n\left|\frac{2\pi-s-t}{s-t}\right|\right)(\pi-s)^{-1}ds + O(\pi-t)$$

$$\le \frac{1}{3\pi}\int_0^{(\pi-\delta)} \left(\ell n\left|\frac{2\pi-s-t}{s-t}\right|\right)(\pi-s)^{-1}ds + O(\pi-t) \quad .$$

Therefore, as is shown in the proof of Theorem 4.1,

$$V(\delta;t) \leq \frac{\pi}{6} + O(\pi-t), \quad t \in (0,\pi) \quad , \tag{5.5}$$

while

$$V(\delta;t) \leq \text{const.}\frac{(\pi-t)}{\delta}, \quad t \in (\pi-\delta,\pi) \quad , \tag{5.6}$$

the constant being independent of δ. The use of (5.4) - (5.6) in (5.3) gives

$$\int_{\pi/2}^{\pi-\delta} \frac{\theta(s)}{(\pi-s)} ds \leq \text{const.} + \int_{\pi/2}^{\pi} V(\delta;t) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt$$

$$\leq \text{const.} + \frac{\pi}{6}\int_{\pi/2}^{(\pi-\delta)} \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt + \text{const.}\frac{1}{\delta}\int_{(\pi-\delta)}^{\pi} \sin\theta(t)dt$$

$$\leq \text{const.} + \frac{\pi}{6}\ln\left(\int_0^{(\pi-\delta)} f(w)\sin\theta(w)dw\right) - \frac{\pi}{6}\ln\left(\int_0^{\pi/2} f(w)\sin\theta(w)dw\right) ,$$

from which the finiteness of $\int_{\pi/2}^{\pi} f(t)\sin\theta(t)dt$ follows. Now this gives that $\lim_{s\to\pi} \theta(s) = 0$, as in (D), whence it follows that $\int_0^{\pi} f(t)\sin\theta(t)dt \geq \pi/6$ as in Theorem 3.7(c). The upper bound M can also be shown to exist by a routine adaptation of the proof of (E).

(d) By (5.1) and Theorem 2.5(f),

$$\theta(s) = \frac{1}{3\pi}\int_0^{\pi} \left(\ln\left|\frac{s+t}{s-t}\right|\right)\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt +$$

$$+ \frac{2}{3}\int_0^{\pi} \tilde{h}(s,t) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt$$

$$\geq \frac{1}{3\pi}\int_0^{\pi} \left(\ln\left|\frac{s+t}{s-t}\right|\right)\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt - Ms \quad ,$$

where M is a positive constant,

$$\geq \frac{1}{3\pi}\int_0^{\pi}\left(\ln\left|\frac{s+t}{s-t}\right|\right)\frac{\sin\theta(t)}{\int_0^t \sin\theta(w)dw}\,dt - Ms \quad , \tag{5.7}$$

by the footnote on page 48.

$$\geq \frac{1}{3\pi}\int_0^{\pi}\left(\ln\left|\frac{s+t}{s-t}\right|\right)\frac{\sin\theta(t)}{t}\,dt - Ms \quad ,$$

$$= \frac{1}{3\pi}\int_0^{\pi/s}\left(\ln\left|\frac{1+t}{1-t}\right|\right)\frac{\sin\theta(st)}{t}\,dt - Ms$$

$$\geq \frac{1}{3\pi}\int_0^{1}\left(\ln\left|\frac{1+t}{1-t}\right|\right)\frac{\sin\theta(st)}{t}\,dt - Ms$$

$$\geq \text{const.}\int_0^1 \sin\theta(st)dt - Ms$$

$$= \text{const. } s^{-1}\int_0^s \sin\theta(t)dt - Ms \quad .$$

Hence

$$Ms + \theta(s) \geq \text{const. } s^{-1}\int_0^s \sin\theta(t)dt \quad . \tag{5.8}$$

However since $0 < \theta(s)$ on $(0,\pi)$, by Theorem 2.5(c) there exists $\delta > 0$ with $\theta(s) \geq \delta \sin s$ on $(0,\pi)$. Therefore, for all $s \in (0,\pi/2)$, (5.8) ensures that there exists an α such that $\theta(s) \geq \alpha s^{-1}\int_0^s \sin\theta(t)dt$.

(e) Since $\theta(s) \geq \delta \sin s$ for some $\delta > 0$, it follows from (5.7) that for some $\beta > 0$

$$\theta(s) \geq \beta\int_0^{\pi/2}\left(\ln\left|\frac{s+t}{s-t}\right|\right)\frac{\sin\theta(t)}{\int_0^t \sin\theta(w)dw}\,dt$$

for all $s \in (0,\pi/2)$. Therefore, for each $\varepsilon \in (0,\pi/2)$,

$$\frac{1}{\varepsilon}\int_0^{\varepsilon}\theta(s)ds \geq \frac{\beta}{\varepsilon}\int_0^{\varepsilon}\{\int_0^{\pi/2}(\ln\left|\frac{s+t}{s-t}\right|)\frac{\sin\theta(t)}{\int_0^t \sin\theta(w)dw}dt\}ds$$

$$= \beta\int_0^{\pi/2} W(\varepsilon;t)\frac{\sin\theta(t)}{\int_0^t \sin\theta(w)dw}dt \quad , \tag{5.9}$$

where

$$W(\varepsilon;t) = \frac{1}{\varepsilon}\int_0^{\varepsilon}(\ln\left|\frac{s+t}{s-t}\right|)ds$$

$$= \frac{t}{\varepsilon}\int_0^{\varepsilon/t}(\ln\left|\frac{1+s}{1-s}\right|)ds$$

$$\geq \frac{t}{\varepsilon}\left(\frac{\varepsilon}{t}\right)^2 = \frac{\varepsilon}{t}$$

for all $t \in (\varepsilon,\pi)$. Hence, by (5.9)

$$\frac{1}{\varepsilon}\int_0^{\varepsilon}\theta(s)ds \geq \beta\,\varepsilon\int_{\varepsilon}^{\pi/2}\frac{\sin\theta(t)}{t\int_0^t \sin\theta(w)dw}dt \quad ,$$

for all $\varepsilon \in (0,\pi/2)$.

(f) By (b) and (d)

$$s\sin\theta(s) \geq \frac{2\alpha}{\pi}\int_0^s \sin\theta(t)dt \quad ,$$

for all $s \in (0,\pi/2)$, and so it follows from (e) that for $\varepsilon \in (0,\pi/2)$,

$$\frac{1}{\varepsilon}\int_0^{\varepsilon}\theta(s)ds \geq \frac{2\alpha\beta\varepsilon}{\pi}\int_{\varepsilon}^{\pi/2}\frac{1}{t^2}dt$$

$$= \frac{2\alpha\beta}{\pi}\left(1 - \frac{2\varepsilon}{\pi}\right) \quad ,$$

whence the result follows, by (d).

(g) To begin with, we assume that

$$\liminf_{s\to 0+}\frac{1}{s}\int_0^s \sin\theta(t)dt \geq \frac{1}{2} \quad . \tag{5.10}$$

Then for any $\varepsilon > 0$ sufficiently small, there exists $t_\varepsilon > 0$ such that

$$\int_0^t f(w)\sin\theta(w)dw \geq (\frac{1}{4} - \varepsilon)t$$

and

$$f(t)\sin\theta(t) \leq \frac{1}{2}\sin\alpha + \varepsilon \quad ,$$

for all $t \in (0,t_\varepsilon)$, where $\alpha = \limsup_{s\to 0+} \theta(s) > 0$, by (f). Therefore if $t \in (0,t_\varepsilon)$,

$$\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} \leq \left(\frac{\frac{1}{2}\sin\alpha+\varepsilon}{\frac{1}{4}-\varepsilon}\right)\frac{1}{t} \quad . \tag{5.11}$$

By Theorem 2.5(f),

$$\theta(s) = \frac{1}{3\pi}\int_0^\pi \left(\ln\left|\frac{s+t}{s-t}\right|\right) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt + \frac{2}{3}\int_0^\pi \tilde{h}(s,t) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt$$

$$= \frac{1}{3\pi}\int_0^{t_\varepsilon} \left(\ln\left|\frac{s+t}{s-t}\right|\right) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt + \frac{1}{3\pi}\int_{t_\varepsilon}^\pi \left(\ln\left|\frac{s+t}{s-t}\right|\right) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt$$

$$+ O(s) \quad .$$

Clearly, for each $\varepsilon > 0$ and corresponding $t_\varepsilon > 0$,

$$\lim_{s\to 0+} \frac{1}{3\pi}\int_{t_\varepsilon}^\pi \left(\ln\left|\frac{s+t}{s-t}\right|\right) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt = 0 \quad ,$$

and so

$$\alpha = \limsup_{s\to 0+} \frac{1}{3\pi}\int_0^{t_\varepsilon} \left(\ln\left|\frac{s+t}{s-t}\right|\right) \frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} dt$$

$$\leq \left\{\frac{\frac{1}{2}\sin\alpha+\varepsilon}{\frac{1}{4}-\varepsilon}\right\}\limsup_{s\to 0+}\frac{1}{3\pi}\int_0^{t_\varepsilon}\left(\ln\left|\frac{s+t}{s-t}\right|\right)\frac{1}{t}\,dt ,$$

by (5.11),

$$= \left\{\frac{\frac{1}{2}\sin\alpha+\varepsilon}{\frac{1}{4}-\varepsilon}\right\}\limsup_{s\to 0+}\frac{1}{3\pi}\int_0^{t_\varepsilon/s}\left(\ln\left|\frac{1+t}{1-t}\right|\right)\frac{1}{t}\,dt$$

$$= \frac{\pi}{6}\left\{\frac{\frac{1}{2}\sin\alpha+\varepsilon}{\frac{1}{4}-\varepsilon}\right\} . \qquad (5.12)$$

Consequently, $0 < \alpha \leq \frac{\pi}{3}\sin\alpha$, and it is easy to show from this that $\alpha \leq \pi/6$. It follows that

$$\limsup_{s\to 0+}\frac{1}{s}\int_0^s \sin\theta(t)\,dt \leq \frac{1}{2} ,$$

which, with (5.10) gives that

$$\lim_{s\to 0+}\frac{1}{s}\int_0^s \sin\theta(t)\,dt = \frac{1}{2} .$$

Set $\beta = \liminf_{s\to 0+}\theta(s)$. Then $\beta > 0$ by (f), and an argument as for (5.12) yields that for all $\varepsilon > 0$ sufficiently small,

$$\beta \geq \frac{\pi}{6}\left\{\frac{\frac{1}{2}\sin\beta-\varepsilon}{\frac{1}{4}+\varepsilon}\right\} ,$$

whence $\beta \geq \frac{\pi}{3}\sin\beta > 0$, and $\beta \geq \pi/6$. This shows that if (5.10) holds then $\lim_{s\to 0+}\theta(s) = \frac{\pi}{6}$.

Now suppose that

$$\limsup_{s\to 0+}\frac{1}{s}\int_0^s \sin\theta(t)\,dt \leq \frac{1}{2} .$$

If $\varepsilon > 0$ is sufficiently small, then there exists $t_\varepsilon > 0$ with

$$\frac{f(t)\sin\theta(t)}{\int_0^t f(w)\sin\theta(w)dw} \geq \{\frac{\frac{1}{2}\sin\beta-\varepsilon}{\frac{1}{4}+\varepsilon}\}t$$

for $t \in (0,t_\varepsilon)$. As before, it follows that $\beta \geq \frac{\pi}{3}\sin\beta > 0$, whence $\beta \geq \frac{\pi}{6}$. Hence (5.10) holds and the result follows as before. q.e.d.

We have already emphasized the distinction between the question of the existence of a solitary wave of greatest height, and the question of the existence of a solitary wave which satisfies Stokes' conjecture. When Stokes ([43; vol. I, page 227]) considered the possibility of a limiting wave, he felt that the question of the limiting form near the crest was settled and that doubt remained only concerning the existence of such limiting waves. After arguing that the wave of greatest height is sharp-crested, and that the separate tangents to the free surface at the crest subtend an angle of 120° with each other, he wrote

"This however leaves untouched the question whether the disturbance can actually be pushed to the extend of yielding crests with sharp edges, or whether on the other hand there exists a limit, for which the outline is still a smooth curve, beyond which no waves of oscillatory irrotational type can be propagated without change of form.

After careful consideration I feel that there is no such early limit, but that we may actually approach as near as we please to the form in which the curvature is infinite, and the vertex becomes a multiple point where the two branches with which alone we are concerned enclosed an angle of 120°."

Here Stokes is writing about periodic waves, but his remarks are equally for solitary waves. His surmise in the second paragraph is correct both for the periodic wave [47] and for the solitary wave (our section 5). So far, all that is known about the form of the limiting wave is in Theorem 5.2, and may be summarized in the following dichotomy: either

$$\lim_{s \to 0+} \theta(s) = \frac{\pi}{6} \quad ,$$

or

$$\liminf_{s \to 0+} \theta(s) < \frac{\pi}{6} < \limsup_{s \to 0+} \theta(s) \quad .$$

In other words, either Stokes' conjecture is true, or there are an infinite number of points of inflection on the limiting wave profile in every neighborhood of the crest.

APPENDIX.

ELEMENTS OF THE THEORY OF POSITIVE OPERATORS AND GLOBAL BIFURCATION

If X is a real Banach space, a set $K \subset X$ is call a cone if (i) K is closed, (ii) for $u, v \in K$, $\alpha, \beta > 0$, $\alpha u + \beta v \in K$, and (iii) if $u \in K$ and $-u \in K$ then $u = 0$. If $u-v \in K$, then we write $u \geq v$. A cone is called reproducing if each $u \in X$ can be written as $v - w$ where v and w are in K (clearly such a representation cannot be unique).

If K is a cone in a Banach space X, then a linear operator B from X into itself is called positive if $Bu \in K$ when $u \in K$. If u_0 is a non-zero element of the cone K and for each non-zero $u \in K$

$$B^n u \geq \varepsilon u_0 \quad ,$$

for some integer n and positive number ε (both of which may depend on u), then B is said to be u_0-bounded below with respect to K. Analogously B is u_0-bounded above if for each $u \in K$, there exists a positive integer m and $M > 0$ such that

$$B^m u \leq M u_0 \quad ,$$

where, once again, m and M may depend on u. An operator which is both u_0-bounded above and below is called u_0-positive.

As usual, a continuous operator on X is called completely continuous if the image of every bounded set is relatively compact. Let $c_K(B) = \{\mu \geq 0 : u = \mu Bu$ for some non-trivial $u \in K\}$. Then $c_K(B)$ is the set of positive characteristic values of B, where a positive characteristic value is defined to be the reciprocal of an eigenvalue which has an eigenvector in K.

THEOREM A1. (A special case of Theorem 2.5 [22]) Let $B : X \to X$ be a completely continuous linear operator which is positive with respect to a cone K. If for some non-zero $u \in K$ and $\alpha > 0$,

$$\alpha Bu \geq u \quad ,$$

then $c_K(B) \cap (0,\alpha] \neq \phi$.

THEOREM A2. (Theorems 2.10, 2.11, 2.13 [22]) Let $\Psi_0 \in K$ be an eigenvector of the linear operator $B : X \to X$, which is u_0-positive with respect to the reproducing cone K. Then

(a) Ψ_0 is the unique normalized eigenvector of B in K,

(b) the corresponding eigenvalue λ_0 is simple, and

(c) λ_0 is greater than the absolute value of the remaining eigenvalues of B.

THEOREM A3. Let $B : X \to X$ be a completely continuous linear operator which is u_0-positive with respect to the reproducing cone K. Then there exists $(\lambda_0, \psi_0) \in [0,\infty) \times K$ such that Theorem A2 holds. Moreover λ_0 is the spectral radius of B and $c_K(B) = \{\lambda_0^{-1}\}$.

Proof. Since B is completely continuous, the only non-zero points in its spectrum are eigenvalues of finite multiplicity. The result follows from Theorem A1 and A2. q.e.d.

In the discussion of nonlinear positive operator equations that follows, K *is a reproducing cone* in the Banach space X with norm $|\cdot|$. We will consider operator equations of the form

$$u = A(\mu,u) \tag{a1}$$

where $A : [0,\infty) \times K \to K$ and has the following properties:

(i) A is continuous and maps bounded subsets of $[0,\infty) \times K$ into relatively compact subsets of X.

(ii) $A(\mu,0) = A(0,u) = 0$ for all $(\mu,u) \in [0,\infty) \times K$.

(iii) $A(\mu,u) = \mu Bu + F(\mu,u)$ for all $(\mu,u) \in [0,\infty) \times K$, where $B : X \to X$ is a positive linear operator as in Theorem A3, and

$$|F(\mu,u)|/|u| \to 0$$

as $|u| \to 0$ uniformly for μ in bounded intervals of $[0,\infty)$.

Let (λ_0,ψ_0) be given as in Theorem A3, and set $S = \{(\mu,u) \in [0,\infty) \times (K\setminus\{0\}) : u = A(\mu,u)\} \cup \{(\lambda_0^{-1},0)\}$. Then this next result follows as a corollary of Theorem 2 [14] (see also [48]).

THEOREM A4. Under all the assumptions on A listed above, the maximal connected subset C of S which contains $(\lambda_0^{-1},0)$ is closed and unbounded in $[0,\infty) \times X$.

Proof. That C is unbounded is immediate by the corollary to Theorem 2 [14] and the fact that $c_K = \{\lambda_0^{-1}\}$. To show that C is closed, it suffices to consider a sequence $\{(\mu_n,u_n)\} \subset C$ such that $(\mu_n,u_n) \to (\mu,u)$ and to show that $(\mu,u) \in C$. The continuity of A ensures that $(\mu,u) \in [0,\infty) \times K$ and that $u = A(\mu,u)$. If $u = 0$, then a simple argument shows that $|u_n|^{-1}u_n \to v$ in X, where $v = \mu Bv$, and $v \in K$ with $|v| = 1$. Hence, by Theorem A3, $\mu = \lambda_0^{-1}$ and $(\mu_n,u_n) \to (\lambda_0^{-1},0) \in C$. q.e.d.

The following result of Whyburn [49] allows global bifurcation to be described in terms of open sets in $[0,\infty) \times K$ (Theorem A6 below).

THEOREM A5. Suppose that F_1 and F_2 are non-empty closed subsets of a compact metric space M and suppose that no connected subset of M has non-empty intersection with both F_1 and F_2. Then there exist compact subsets M_1 and M_2 of M with $F_1 \subset M_1$, $F_2 \subset M_2$; $M_1 \cap M_2 = \phi$, and $M_1 \cup M_2 = M$.

THEOREM A6. Let X be a Banach space and let $S \subset \mathbb{R} \times X$ be closed, and such that its bounded subsets are relatively compact. If $(\lambda_0,0) \in S$, then the maximal connected subset C of S which contains $(\lambda_0,0)$ is closed and unbounded if and only if $\partial U \cap S \neq \phi$ for every bounded, open subset U in $\mathbb{R} \times X$ with $(\lambda_0,0) \in U$.

Remark. It will be clear from the proof that if C is unbounded, then $\partial U \cap S \supset \partial U \cap C \neq \phi$, whether bounded subsets of S are relatively compact or not. This latter hypothesis on S is required only for the sufficiency point of the proof.

Proof of Theorem A6. Suppose that C is unbounded and that U is as above. Then $(\lambda_0,0) \in U \cap C \neq \phi$ and $((\mathbb{R} \times X)\backslash \bar{U}) \cap C \neq \phi$. Since C is connected, $\partial U \cap C \neq \phi$ and a fortiori $\partial U \cap S \neq \phi$.

Let us assume now that C is bounded, yet $\partial U \cap S \neq \phi$ for every bounded, open set U with $(\lambda_0,0) \in U$. For some $R > 0$, $C \subset B(R)$, where $B(R) = \{(\lambda,u) : |\lambda| + |u| < R\}$. Let M denote the compact metric space $\overline{B(2R)} \cap S$ (a metric subspace of the Banach space $\mathbb{R} \times X$), and put $F_1 = C$ and $F_2 = \{(\lambda,u) \in S : |\lambda| + |u| = 2R\}$. Now F_1 and F_2 are non-empty, disjoint, compact subsets of M (C is closed, being a maximal, connected subset of the closed set S, and $F_2 \neq \phi$ since $S \cap \partial B(2R) \neq \phi$).

Since C is maximal, Theorem A5 applies and there exist disjoint compact subsets M_1 and M_2 of M such that $M_1 \cup M_2 = M$, $C \subset M_1$ and $S \cap \partial B(2R) \subset M_2$. Note that $\operatorname{dist}(M_1,\partial B(2R)) > 0$, and put $\delta = \min\{\operatorname{dist}(M_1,M_2), \operatorname{dist}(M_1,\partial B(2R))\} > 0$. Now set $U = \{(\lambda,u) \in \mathbb{R} \times X : |\lambda-\lambda'| + |u-u'| < \delta/2$ for $(\lambda',u') \in M_1\}$. Then U is open, bounded, and $(\lambda_0,0) \in U$, whence there exists $(\lambda,u) \in \partial U \cap S$. However, the choice of δ implies that $(\lambda,u) \in M$ and $(\lambda,u) \notin M_1$. Hence

$(\lambda,u) \in M_2$. This is also impossible since $\text{dist}\{(\lambda,u),M_1\} \leq \delta/2$ and $\text{dist}\{M_1,M_2\} \geq \delta$. With this contradiction, we conclude that C is unbounded. It has already been observed that C is closed, being a maximal connected subset of the closed set S. q.e.d.

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